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ON CERTAIN LOCALLY FLAT HOMOGENEOUS MANIFOLDS OF SOLVABLE LIE GROUPS

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Introduction

Let M be a connected differentiable manifold with a locally flat linear connection D (A linear connection is locally flat, if its torsion and curvature tensors vanish identically). Then, for each point $p \in M$, there exists a local coordinate system $\{x^1, \cdots, x^n\}$ in a neighbourhood of p such that $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$, which we call an *affine local coordinate system.* A Riemannian metric g on M is said to be *locally Hessian* with respect to D, if for, each point $p \in M$, there exists a real-valued function ϕ of class C^∞ on a neighbourhood of p such that

$$g = D^2 \phi ,$$

that is,

$$g = \sum_{i,j} \frac{\partial^2 \phi}{\partial x^i \partial x^j} dx^i dx^j ,$$

where $\{x^1, \cdots, x^n\}$ is an affine local coordinate system around p. If this condition is verified with a function ϕ defined over M, the metric g is called a *Hessian metric* on M. A locally flat manifold with a (locally) Hessian metric is called a (*locally*) *Hessian manifold.*

The following proposition is essentially due to S. Murakami and will be proved in §1.

Proposition. *Let M be a connected differentiable manifold with a locally flat linear connection D and a Riemannian metric g. Let γ be the cotangent bundle-valued* 1*-form on M defined by*

$$(\gamma(X))(Y) = g(X, Y)$$

for vector fields X, Y on M. The cotangent bundle being locally flat, we may consider the exterior differentiation $\underline{\underline{d}}$ for cotangent bundle-valued forms on M. Then the following conditions (1)$\sim$(4) *are equivalent*:

(1) *g is locally Hessian with respect to D.*