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A NOTE ON DIRECT SUMS OF CYCLIC MODULES OVER COMMUTATIVE REGULAR RINGS

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Throughout this note R is a commutative ring with identity and all modules are unital.

We denote the maximal ring of quotients of R by $Q(R)$ and the ring generated by the set of all idempotents of $Q(R)$ over R by $C(R)$. In case R is semi-prime, $C(R)$ coincides with the Baer hull of R in the sense of *A. C.* Mewborn ([3, Proposition 2.5]). For an R-module M, we denote its injective hull by $E_R(M)$. It is well known (e.g. [1]) that if R is semi-prime, then $Q(R)=E_R(R)$.

Let M be an R-module. We put

$$\begin{aligned} T(M) &= \{x\in M \mid \mathrm{Hom}_R(Rx, E_R(R)) = 0\} \\ &= \{x\in M \mid (0:x) \text{ is a dense}^{\dagger} \text{ ideal of } R\} \end{aligned}$$

where $(0:x)=\{r\in R \mid rx=0\}$ (see [7]). M is said to be torsion if $T(M)=M$ and torsion free if $T(M)=0$.

Now, for an R-module M, we shall consider the following condition studied in [4]:

(*) M is embedded in a direct sum of cyclic R-modules as an essential R-submodule.

In [4] the author proved the following

Theorem 1. *Let R be a regular ring. Then the following conditions are equivalent*:

(a) $Q(R)=C(R)$.

(b) *Every finitely generated torsion free R-module M satisfies the condition* (*)

The purpose of this note is to prove the following two theorems.

Theorem 2. *Let R be a regular ring. Then the following conditions are equivalent*:

(a) $Q(R/I)=C(R/I)$ *for every dense ideal I of R.*

† An ideal I of R is said to be dense provided that for any r and r' in R with $r'\neq 0$, there exists s in R such that $sr\in I$ and $sr'\neq 0$.