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ON PERFECT RINGS AND THE EXCHANGE PROPERTY

Dedicated to Professor Kiiti Morita on his 60th birthday

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Let R be a ring with unit element. We always consider unitary right R-modules. Let T be an R-module and η a cardinal number. If for any module K containing T as a direct summand and for any decomposition of K with η components: $K=\sum_{\alpha\in I}\oplus A_\alpha$, there exist submodules A_α' of A_α for all a such that $K=T\oplus\sum_{\alpha\in I}\oplus A_\alpha'$, then we say T has the *η-exchange property* [2]. If T has the η-exchange property for any η, we say T has the *exchange property*.

In this short note we shall show that R is a right perfect ring if and only if for every projective module P, P has the exchange property and $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ is a regular ring in the sense of Von Neumann. This is a refinement of Theorem 7 in [4] and we shall give its proof as an application of [6].

After submitting this paper to the journal, the authors have received a manuscript of Yamagata [13] and found that one of main theorems in this paper overlaped with one in [13]. The authors would like to express their thanks to Dr. Yamagata for his kindness.

1. Preliminaries

First we shall recall some definitions given in [3], [4] and [6]. Let T be an R-module. If $\mathrm{End}_R(T)$ is a local ring, T is called *completely indecomposable*. We take a set $\{M_\alpha\}_I$ of completely indecomposable modules and define the full additive subcategory $\mathfrak{A}$ of all right R-modules which is induced from $\{M_\alpha\}_I$, namely the objects in $\mathfrak{A}$ consist of all modules which are isomorphic to direct-sums of completely indecomposable modules in $\{M_\alpha\}_I$. We define an ideal $\mathfrak{J}'$ in $\mathfrak{A}$ as follows: let $A=\sum_{\alpha\in K}\oplus A_\alpha$, $B=\sum_{\beta\in L}\oplus B_\beta$ be in $\mathfrak{A}$, where A_α, B_β are isomorphic to some in $\{M_\alpha\}_I$, then $\mathfrak{J}'\cap[A, B]=\{f\,|\in\mathrm{Hom}_R(A, B), p_\beta f i_\alpha$ are not isomorphic for all $\alpha\in K, \beta\in L\}$, where $i_\alpha: A_\alpha\rightarrow A, p_\beta: B\rightarrow B_\beta$ are the inclusion and the projection, respectively. By $\bar{\mathfrak{A}}$ we denote the factor category of $\mathfrak{A}$ with respect to $\mathfrak{J}'$ [3]. For any object A and morphism f in $\mathfrak{A}$, by $\bar{A}$ and $\bar{f}$ we denote the residue classes of A and f in $\bar{\mathfrak{A}}$, respectively.