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PRODUCTS OF TORSION THEORIES AND APPLICATIONS TO COALGEBRAS

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1. Introduction

Throughout this note R is a ring with 1. We shall write $/ \leqslant R$ if / is a right ideal of R. A non-empty set of right ideals Γ of R is called a Gabriel filter if it satisfies

T1. If $I \in \Gamma$ and $r \in R$, then $(I:r) \in \Gamma$.

T2. If / is a right ideal and there exists $J \in \Gamma$ such that $(I:r) \in \Gamma$ for every $r \in J$, then $I \in \Gamma$.

It is well-known [4] that there is a one to one correspondence between Gabriel filters of R and hereditary torsion theories for the category of right R-modules. W. Schelter [3] investigated products of torsion theories or equivalently of Gabriel filters that for a family of pairs $\{(R_i, \Gamma_i), \Gamma_i:$ Gabriel filter of $R_i\}$, $\Gamma_0 = \{D \leqslant \pi R_i \mid D \supseteq \sum_{\oplus} D_i, D_i \in \Gamma_i\}$ is a Gabriel filter of the product ring πR_i, furthermore the ring of right quotient of πR_i with respect to Γ_0 is isomorphic to the product of rings of right quoteint of R_i with respect to Γ_i: $(\pi R_i)_{\Gamma_0} \cong \pi (R_i)_{\Gamma_i}$ This result generalizes one of Y. Utumi theorems [6]. In this paper these two sets $\Gamma_1 = \{D \leqslant \pi R_i \mid D \supseteq \pi D_i, D_i \in \Gamma_i\}$ and $\Gamma_2 = \{D \leqslant \pi R_i \mid D \supseteq \pi D_i D_i \in \Gamma_i$ and almost all $D_i = R_i\}$ will be studied. Γ_1 does not always satisfy T2. A necessary and sufficient condition for Γ_1 to be a Gabriel filter is given. It follows that Γ_1 is a better notion of products of perfect torsion theories. However Γ_2 is a Gabriel filter of πR_i, and we use this fact to prove that over an algebraically closed field, cocommutative coalgebra has a torsion rat functor if and only if each space of primitives of its irreducible components is finitedimensional.

For a coalgebra (C, Δ, ε) over a field K, there exists a natural algebra structure on its dual space $C^* = \mathrm{Hom}_K$ (C, K) induced by the diagonal map Δ and every left comodule (M, ϕ_M) over C can be defined as a right C^*-module by $mc^* = (c^* \otimes 1)\phi_M(m), m \in M, c^* \in C^*$. Moroever a right C*-module M is called a rational module if it is a left comodule (M, ϕ_M) over C and its right C^*-module structure is derived in the way described above. With these observations we can embed the category of left C-comodules ${}^C\mathcal{M}$, as a full subcategory, into the category of right C*-modules $\mathcal{M}_{C^*}$. A subspace / of C* is callecd **cofinite**