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ON MATSUSHIMA'S FORMULA FOR THE BETTI NUMBERS OF A LOCALLY SYMMETRIC SPACE

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1. Introduction. Let G be a connected semi-simple Lie group with finite center and with no connected normal, compact subgroups. Let $K\subset G$ be a maximal compact subgroup and let $\Gamma \subset G$ be a discrete subgroup acting freely on $X=G/K$ and so that $\Gamma\backslash G$ is compact. Let $M=\Gamma\backslash X$ then M is a typical compact locally symmetric space of negative curvature. Let G *act* by the right regular action, π_Γ, on $L^2(\Gamma\backslash G)$. In Matsushima [13] a formula for the Betti numbers of M is given in terms of the multiplicities of certain unitary representations of G in π_Γ. In this paper we investigate the existence of the representations of G that come into the Matsushima formula. In particular we show that if X is irreducible and Hermitian symmetric and if rank $X>p$ then there are no unitary representations of G that satisfy the conditions for the Matsushima formulas for the $(0, p)$ Betti number of $\Gamma\backslash X$, $b_{0,p}(\Gamma\backslash X)$. Thus we find that if $p<$rank X, $b_{0,p}(\Gamma\backslash X)=0$. In particular if rank $X>1$ we recover Matsushima's theorem [12] that the first Betti number of $\Gamma\backslash X$ is zero.

Actually this result (on the first Betti number) follows from the more general theorem of Kazhdan which says if G is a simple Lie group of split rank larger than 1 and if $\Gamma\subset G$ is a discrete subgroup so that $\Gamma\backslash G$ has finite volume relative to some Haar measure on G then $\Gamma/[\Gamma, \Gamma]$ is finite.

In light of the above result of Kazhdan it is reasonable to study the unitary representations that appear in the formula of Matsushima for the first Betti number in the case G has split rank 1. We show that in this case such representations always exist. We show that there are at most two such unitary representations and if G is locally $SO(n, 1)$ there is exactly one, let us call it π_1 (see Lemma 2.1, Prop. 2.2 and Lemma 4.4). This gives us some interesting examples. E. B. Vinberg [16] has constructed a uniform discrete subgroup $\Gamma\subset SO_e(n,1)$ for $3\leq n\leq 5$, which is arithmetic in the sense of Borel, Harish-Chandra, such that $\Gamma/[\Gamma, \Gamma]$ is infinite. In Johnson-Wallach [7] it is shown that π_1 cannot be tempered in the sense of Harish-Chandra (see G. Warner [14]) if $n>4$. Thus there exists an arithmetic uniform discrete subgroup Γ of a simple algebraic group G and a non-trivial non-tempered representation with positive multi-