Suzuki, T.
Osaka J. Math.
11 (1974), 653-661

# THE SINGULARITY OF INFINITE PRODUCT MEASURES

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(Recieved January 28, 1974)

**1. Introduction.** In this paper we give a consideration on singularity and nonsingularity relations between two infinite direct product measures. In statistical asymptotic theory the concept of singularity is closely related to the existence of consistent sequence of test procedures. L. Shepp [4] considered such a problem in the case where $P_t$, the component distribution with a location parameter $t$ of the product measure, has a constant carrier and the Fisher's information number exists and is finite. Later L. LeCam [3] extended the result of L. Shepp to the case where $P_t$ satisfy the quadratic mean differentiability conditions. Our purpose is to seek conditions under which given two product measures are singular, when the components $P_t$ do not necessarily satisfy such regular conditions.

In Section 2 we introduce new quantities $\bar{I}$ and $\underline{I}$, and using them conditions of singularity are described. Section 3 is devoted to the proof of the quantity $\bar{I}$ being an extension of the Fisher's information number. In Section 4 we consider the relation between the concept of nonsingularity and that of contiguity which was introduced in L. LeCam [2]. Two examples not satisfying the usual conditions are given in Section 5.

**2. The condition of singularity.** Let $\Theta$ be an open set containing zero. For each $t\in\Theta$ let $P_t$ be a probability measure on a certain $\sigma$-field $\mathfrak{A}$ of subsets of a set $X$. Let $(X^N, \mathfrak{A}^N)$ be the Cartesian product of countably many copies of $(X, \mathfrak{A})$. Let $Q_0=\prod_{i=1}^{\infty} P_0^{(i)}$ (direct product), $P_0^{(i)}=P_0\ (i=1, 2, \cdots)$ and $Q_\pi=\prod_{i=1}^{\infty} P_{h_i}$ for $\pi=(h_1, h_2, \cdots)$. We say that $Q_0$ and $Q_\pi$ are singular if there exists a set $B$ in $\mathfrak{A}^N$ such that $Q_0(B)=0$ and $Q_\pi(B)=1$. In this section conditions are given to the sequence $\pi$ for which $Q_0$ and $Q_\pi$ are singular. Let $H=\{h; h\geqq 0, h\in\Theta, P_0 \text{ and } P_h \text{ are not singular.}\}$ and $\bar{H}=\prod_{i=1}^{\infty} H^{(i)}$, $H^{(i)}=H\ (i=1, 2, \cdots)$. In the following the lower-case letters $h$ with or without suffixes always mean the elements taken from the set $H$. Throughout this paper the following assumptions (A-1) and (A-2) will be made.

(A-1) $\{P_t; t\in\Theta\}$ is dominated by a $\sigma$-finite measure $\mu$ on $X$.