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ON SYMMETRIC STRUCTURE OF A FINITE SET

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1. Introduction

A symmetric structure of a finite set A is defined to be a mapping S of A into the group of permutations on A (the image of an element a in A by S is denoted by S_a or by $S[a]$ and the image of an element b in A by a permutation S_a is denoted by bS_a) such that (i) $aS_a=a$, (ii) $S_a^2=I$ (the identity permutation) and (iii) $S[bS_a]=S_aS_bS_a$ for a and b in A. A set with a symmteric structure is called a symmetric set (with a given symmetric structure). Every group G has a symmetric structure S defined by $bS_a=ab^{-1}a$ for a and b in G, and when we regard a group as a symmetric set we always take this symmetric structure. Generally a symmetric set has a more complicate structure than a group and to develop a structure theory of a symmetric set seems to be an open problem. In this note, we first investigate the following two conditions.

(E) $S_a \neq S_b$ if $a \neq b$.

(H) For any elements a and b, there exists an element c such that $aS_c=b$.

Symmetric sets which satisfy (E) (or (H)) are called *effective* (or *homogeneous*).

Proposition 1. (H) *implies* (E).

Proof. Suppose that (H) is satisfied. Fix an element a and consider a correspondence $b \rightarrow b'$ defined by $aS_b=b'$. The correspondence is a surjective mapping of A to A due to (H). Since A is a finite set, it is a bijection. Therefore, if $b \neq c$, then $aS_b \neq aS_c$. Naturally $S_b \neq S_c$.

Actually (H) is stronger than (E).

Example 1. Let $A=\{1, 2, 3, 4, 5, 6\}$. Consider S defined by $S_1=(24)(36)$, $S_2=(14)(35)$, $S_3=(25)(16)$, $S_4=(56)(12)$, $S_5=(23)(46)$ and $S_6=(45)(13)$. S is a symmetric structure of A. (E) is satisfied but not (H), since 1 is not mapped to 4 by any S_i.

Next, we consider the group of displacements of A, which is defined to be a subgroup of the group of permutations on A generated by S_aS_b for all a and b in A. Denote it by $G(A)$.