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COBORDISM ALGEBRA OF METACYCLIC GROUPS

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1. Introduction and the statement of result

Let $Z_{q,p}$ be *the metacyclic group* $\{x, y \mid x^q=y^p=1, yxy^{-1}=x^r\}$ where $p\geqslant 2$ is a prime integer, $q\geqslant 3$ is an odd integer and r is a primitive p-th root of 1 mod q such that $(r-1, q)=1$ (see Shibata [15], 1. Introduction). The object of the present article is to determine the oriented and weakly complex cobordism algebra $\Omega_L^*(Z_{q,p})$ $(L=SO, U)$ of the classifying space of $Z_{q,p}$. Our main result is the following.

Theorem. *Let $p\geqslant 3$ in case $L=SO$ and $p\geqslant 2$ in case $L=U$.*
(1) *$\Omega_L^*(Z_{q,p})$ is a sum of two Ω_L^*-subalgebras whose intersection is the scalars Ω_L^*. These subalgebras are the quotients of power series rings over Ω_L^* generated respectively by the cobordism Euler classes $e(\nu_{q,p})$ and $e(\tilde{\eta}_p)$.*

Here $\tilde{\eta}_p$ is the pull-back of the Hopf line bundle η_p over B_{Z_p} and $\nu_{q,p}$ is a certain complex vector bundle of dimension p whose "restriction" on B_{Z_q} is $\eta_q\oplus\eta_p^r\oplus\cdots\oplus\eta_q^{r^{p-1}}$ with η_q the Hopf line bundle over B_{Z_q}.

(2) $\Omega_L^*(Z_{q,p})\cong\Omega_L^*[[(\prod_{j=0}^{p-1} [r^j]_F(X)), \ Y]]/(XY, \ [q]_F(X), [p]_F(Y), \ (\text{Tor } \Omega_L^*)X,$ $(\text{Tor } \Omega_L^*)Y)$, *where $[\ ]_F$ indicates the formal multiplication with respect to the formal group law $F(X, Y)$ of complex cobordism theory (or its canonical reduction to oriented cobordism for $L=SO$) (see Quillen* [12]), Tor $\Omega_U^*=0$ *and* Tor Ω_{SO}^* *consists of elements of order* 2. *This isomorphism is realized by the correspondence*; $e(\nu_{q,p})\mapsto$ $\prod_{j=0}^{p-1} [r^j]_F(X)$ *and* $e(\tilde{\eta}_p)\mapsto Y$.

(3) *In case $L=SO$ and $p=2$, the Ω_L^*-subalgebra generated by $e(\tilde{\eta}_p)$ in* (1) *is replaced by $\pi^*\Omega_{SO}^*(Z_2)$, where π^* is the monomorphism induced by the projection $\pi\colon B_{Z_{q,2}}\to B_{Z_2}$. And* (2) *is modified as*

$$\Omega_{SO}^*(Z_{q,2})\cong\Omega_{SO}^*[[\prod_{j=0}^{p-1} [r^j]_F(X)]]/([q]_F(X), (\text{Tor } \Omega_{SO}^*)X)\oplus\tilde{\Omega}_{SO}^*(Z_2)\,.$$

REMARK. $\Omega_{SO}^*[[Y]]/([2]_F(Y))=\Omega_{SO}^*[[Y]]/(2Y)$ is contained in $\Omega_{SO}^*(Z_2)$ as a proper Ω_{SO}^*-subalgebra, Y being the reduction of $e(\eta_2)$ to $\Omega_{SO}^*(Z_2)=\Omega_{SO}^*(B_{Z_2})$. This is easily derived from the results of Shibata [14] via the Atiyah-Poincaré