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ON THE SECTIONAL CURVATURES OF R-SPACES

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Introduction

Let $\mathfrak{g}$ be a real semi-simple Lie algebra without compact factors, $\mathfrak{k}$ a maximal compactly imbedded subalgebra of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ relative to $\mathfrak{k}$. We denote by B the Killing form of $\mathfrak{g}$. We regard the subspace $\mathfrak{p}$ as a Euclidean space with the inner product $\langle\ ,\ \rangle$ induced by the restriction of B to $\mathfrak{p}$. Let Int $(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$ and, the Lie algebra of Int $(\mathfrak{g})$ being identified with $\mathfrak{g}$, K the connected Lie subgroup of Int $(\mathfrak{g})$ corresponding to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Then K leaves the subspace $\mathfrak{p}$ invariant and acts on the Euclidean space $\mathfrak{p}$ as an isometry group. Let S be the unit sphere of $\mathfrak{p}$ and N an orbit of an element H_0 in S. Denoting by K^* the stabilizer of H_0 in K, the space N may be identified with the quotient space K/K^* and is called an *R-space*. We always assume that dim $N\geqq 2$, and the space N is *substantial*, i.e. there exist no proper subspaces of $\mathfrak{p}$ containing N. The aim of this paper is to study the sectional curvatures of N with respect to the K-invariant Riemannian metric $\langle\ ,\ \rangle$ induced by the inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$.

It is known (Takeuchi-Kobayashi [8]) that if the pair (K, K^*) is a symmetric pair, the metric $\langle\ ,\ \rangle$ on $N=K/K^*$ coincides with the K-invariant Riemannian metric defined by a K-invariant inner product on $\mathfrak{k}$, and so the sectional curvatures of N are always non-negative, and N has a positive sectional curvature along each plane section if and only if the pair (K, K^*) is of rank 1.

In this paper we shall show that in general cases the space N may have both positive and negative sectional curvatures. Indeed, the curvatures are related with the restricted root system $\mathfrak{r}$ of $\mathfrak{g}$. Let Δ be a fundamental root system of $\mathfrak{r}$. Then a subsystem Δ_1 of Δ corresponds to the space N (See section 3), and we have:

(I) *If the restricted root system $\mathfrak{r}$ is irreducible and the cardinality $|\Delta-\Delta_1|$ of $\Delta-\Delta_1$ is not less than 2, the space N has both positive and negative sectional curvatures.*

Furthermore we shall characterize the R-spaces with strictly positive sectional