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THE AFFINE STRUCTURES ON THE REAL TWO-TORUS (I)

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Introduction

In these two papers we intend to study the space of all affine structures on the real 2-dimensional torus T^2, a problem suggested by C. Ehresmann in 1936, or more specifically by S. S. Chern in one of his lectures and attacked by N. H. Kuiper [6] among others. An affine structure on a mainfold is a maximal atlas whose coordinate transformations belong to the affine transformation group $A(n)$ on the affine space.

Our main purpose is to describe the set $\{\Gamma\}$ of all affine structures on T^2 module the group $\mathrm{Diff}[T^2]_e$; here $\mathrm{Diff}[T^2]_e$ is the group of all diffeomorphisms of T^2 which induce the identity on the fundamental group $\pi_1(T^2)$. The space $\{\Gamma\}/\mathrm{Diff}[T^2]_e$, equipped with an appropriate topology, is regarded as an affine version of the Teichmüller space.

In the usual case the holonomy group H of an affine structure on a mainfold is defined as a subgroup of the affine transformation group $A(n)$ up to the conjugate class. In this work, however, we construct a modified holomony group H^* for an affine structure so that in the case of 2-dimensional affine torus the group H^* is a subgroup of $\widetilde{A(2)}_e$, the universal covering group of the identity component of $A(2)$. We do this in such a way that the modified holonomy group H^* is mapped onto the usual holonomy group by the projection mapping.

With this modification of the holonomy group the first main result in the paper could be summarized as follows (Theorem 3.3 and 4.15): the affine structures on T^2 are completely determined by their modified holonomy groups H^*.

Carrying out the determination of holonomy groups H^*, we describe the space $\{\Gamma_h\}/\mathrm{Diff}[T^2]_e$ of the homogeneous affine structures on T^2. As Y. Matsushima [7] discusses for complex tori in a somewhat different way, we show the following (Theorem 3.10): the space $\{\Gamma_h\}/\mathrm{Diff}[T^2]_e$ is an affine algebraic or, more precisely a 4-dimensional quadratic cone in $\boldsymbol{R}^6$ without singularities variety, except at the vertex, the vertex itself corresponding to the natural affine

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