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INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER BY QUADRATURES

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Abstract. A differential equation $y'=f(x, y)$ can be solved by quadrature if an infinitesimal transformation $\xi\partial/\partial x+\eta\partial/\partial y$ leaving $y'=f$ invariant is known. This theorem is due to Lie. Here, the converse will be proved in the following form:

Suppose that a one-parameter family of equations $y'=\theta(x, y; a)$ each of which is left invariant by $\xi\partial/\partial x+\eta\partial/\partial y$ is known. Then the equation $\xi dy-\eta dx=0$ can be solved by quadrature.

Through this theorem we shall give a method different from that of Lie for integrating $y'=f(x, y)$ by quadratures.

1. Introduction. Consider a differential equation

$$y' = f(x, y)\,. \tag{1}$$

Suppose that an infinitesimal transformation

$$\xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2}$$

leaves (1) invariant. Then the Pfaffian form

$$(\eta-f\xi)^{-1}(dy-f\,dx)$$

is exactly integrable. This theorem is due to Lie [2, p.97].

Here, we shall consider an infinitesimal contact transformation leaving (1) invariant. Every infinitesimal contact transformation is expressed in the form

$$-\psi_z\frac{\partial}{\partial x}+(\psi-z\psi_z)\frac{\partial}{\partial y}+(\psi_x+z\psi_y)\frac{\partial}{\partial z}\,, \tag{3}$$

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