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# ON FREE ABELIAN EXTENSIONS

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**Introduction**. Let $R$ be a commutative ring, and $G$ a finite abelian group. In [2] (see also [5]) the set of isomorphism classes of Galois extensions of $R$ with group $G$ that have normal bases is described cohomologically by means of Harrison's complex of RG; to this end, Galois algebras are first classified and then Galois extensions with normal basis as a particular case. In this paper we use a different approach to classify Galois extensions which are free as $R$-modules; the restriction of this classification to extensions with normal basis yields the cohomological description of [2].

**Free Abelian Extensions**

Let $R$ be a commutative ring, and $G$ a finite abelian group. Recall that a faithful $R$-algebra $A$ is said to be a Galois extension of $R$ with respect to a representation of $G$ by $R$-algebra automorphisms of $A$ if the following equivalent conditions are satisfied:

1) $A^G=R$ and the map $M_A$ from $A\otimes_R A$ to the ring of functions from $G$ to $A$ defined by $M_A\ (x\otimes y)\ (\sigma)=x\sigma(y)$ is an $R$-module isomorphism.

2) $A^G=R$, $A$ is a finitely generated projective $R$-module and $L{:}AG\rightarrow \mathrm{End}_R(A)$ is an $R$-algebra isomorphism, where $AG$ is the twisted group ring of $G$ over $A$ and $L$ is defined by $L(a\sigma)(x)=a\sigma(x)$.

Let $E$ denote the ring of functions from $G$ to $R$; if we let $G$ act on $E$ by means of $(\sigma f)\ (\eta)=f(\sigma^{-1}\eta)$ then $E$ is Galois over $R$ with group $G$; we have $E=\oplus Re_\sigma$ with $\sum e_\sigma=1$, $e_\sigma e_\eta=\delta_{\sigma,\eta}\ e_\sigma$ and $\sigma(e_\eta)=e_{\sigma\eta}$. Clearly the condition 1) can be reformulated as follows:

3) $A^G=R$ and $M_A$: $A\otimes A\rightarrow E\otimes A$ defined by $M_A(x\otimes y)=\sum e_\sigma\otimes x\sigma(y)$ is an $R$-module isomorphism.

Note that for $M=M_E$: $E\otimes E\rightarrow E\otimes E$ we have $M(e_\alpha\otimes e_\beta)=e_{\alpha\beta^{-1}}\otimes e_\alpha$. Since $EG\simeq \mathrm{End}_R(E)$ we have $EG\otimes EG\simeq \mathrm{End}_R\ (E\otimes E)$; thus considering $E\otimes E$ as a left module over $EG\otimes EG$, the $R$-module automorphisms of $E\otimes E$ are produced by left multiplications by units of $EG\otimes EG$.

Suppose the Galois extension $A$ is free as an $R$-module. Then there exists an $R$-module isomorphism $j$: $A\rightarrow E$ and $M^{-1}\cdot 1\otimes j\cdot M_A\cdot j^{-1}\otimes j^{-1}$: $E\otimes E\rightarrow E\otimes E$ is an isomorphism of $R$-modules. Therefore there exists a unique $u\in U(EG\otimes EG)$