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A NOTE ON THE GROTHENDIECK GROUP OF A FINITE ABELIAN GROUP

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Introduction

For any ring A, by $G(A)$ we denote the Grothendieck group of left A-modules which are finitely generated. Let R be the ring of integers of an algebraic number field K, and let $R\pi$ and $K\pi$ be the group rings of a finite group π over R and K, respectively. If $\mathfrak{O}$ is a maximal R-order in $K\pi$ which contains $R\pi$, then by regarding a module over $\mathfrak{O}$ as one over $R\pi$, we get a homomorphism

$$\psi\colon G(\mathfrak{O}) \to G(R\pi)$$

of Grothendieck groups. Swan [4] proved that ψ is an epimorphism, and Heller and Reiner [2] described the structure of ker ψ by using a map which depends on an ideal theory of the center of $\mathfrak{O}$ and the modular representations of π. The following theorem is an immediate consequence from the description.

Theorem 1. *Let R_i be maximal orders in the center of the simple constituents A_i $(i=1, \cdots, s)$ of $K\pi$. If any prime ideal of R_i which divides the order of π is contained in the ray $J(R_i)$ modulo the real archimedian primes ramified in A_i, then ψ is an isomorphism.*

The purpose of this note is to show that under certain assumptions the converse of this theorem is also true.

Theorem 2. *Let π be a finite abelian group of order n and K be a cyclotomic field. Then ψ is an isomorphism if and only if any prime ideal in $\mathfrak{O}$ which divides n is principal.*

In this case $J(R_i)$ is the group of all principal ideals of R_i and $\mathfrak{O}$ is the direct sum of the R_i. Hence the if part of this theorem is a special case of Theorem 1. Our proof of this theorem is based on a method using the conducter from $\mathfrak{O}$ to $R\pi$, which owes to Swan ([4]).

Throughout this note, modules are assumed to be left modules which are finitely generated and by K, R and $[M]$ we denote an algebraic number field,