Kitamura, K.
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ON THE FREE QUADRATIC EXTENSIONS OF A COMMUTATIVE RING

KAZUO KITAMURA

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Let R be a commutative ring with unit element 1. A quadratic extension of R is an R-algebra which is a finitely generated projective R-module of rank 2. Let $Q(R)$ be the set of all R-algebra isomorphism classes of quadratic extensions of R, and $Q_s(R)$ the set of all R-algebra isomorphism classes of separable quadratic extensions of R. In [2], it was shown that the product in $Q_s(R)$, in the sense of [1], [4] and [5], is extended to $Q(R)$, and $Q(R)$ is an abelian semigroup with unit element. In this note, we study the quadratic extensions of R which are free R-modules. We shall call them the *free quadratic extensions* of R. Let $Q_f(R)$ and $Q_{fs}(R)$ be the sets of all classes which are free R-modules in $Q(R)$ and $Q_s(R)$, respectively. We shall show that $Q_f(R)$ is an abelian semigroup with unit element, and $Q_{fs}(R)$ is an abelian group consisting of all invertible elements in $Q_f(R)$. For some special rings, we shall determine the structures of $Q_f(R)$ and $Q_{fs}(R)$. We remark that $Q_{fs}(R)$, $Q_s(R)$ and $Pic(R)_2$; the group of isomorphism classes $[U]$ of R-module U such that $U\otimes_R U\simeq R$, are closely related by the exact sequence $0 \rightarrow Q_{fs}(R) \rightarrow Q_s(R) \rightarrow Pic(R)_2$.

Let R be any commutative ring with unit element 1. For a free quadratic extension S of R, we can write $S=R\oplus Rx$ and $x^2=ax+b$ for some a, b in R, then we denote it by $S=(R, a, b)$, and by $[R, a, b]$ the R-algebra isomorphism class containing (R, a, b).

Lemma 1. *The following two conditions a) and b) are equivalent;*

a) $(R, a, b)\simeq(R, c, d)$ *as R-algebras,*

b) there exist an invertible element α *in R and an element* β *in R such that* $c=\alpha(a-2\beta)$ *and* $d=\alpha^2(\beta a+b-\beta^2)$.

If (R, a, b) *and* (R, c, d) *satisfy a) or b), then we have*

c) $c^2+4d=\alpha^2(a^2+4b)$ *for some invertible element* α *in R.*

Moreover, if 2 is invertible in R, then we have the converse.

Proof. *a)* $\rightarrow$ *b)*: Let σ: $(R, a, b)=R\oplus Rx \rightarrow (R, c, d)=R\oplus Ry$ be an R-algebra isomorphism, and set $\sigma(x)=\alpha y+\beta$ and $\sigma^{-1}(y)=\alpha'x+\beta'$. Sinec $y=\sigma\cdot\sigma^{-1}(y)=\alpha'\alpha y+\alpha'\beta+\beta'$, we have $\alpha'\alpha=1$, that is, α and α' are invertible. The equalities $(\sigma(x))^2=(\alpha y+\beta)^2=\alpha(\alpha c+2\beta)y+\alpha^2 d+\beta^2$ and $\sigma(x^2)=\sigma(ax+b)=\alpha ay$