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A THEOREM ON FINITE CHEVALLEY GROUPS

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Introduction

In 1955, J.A. Green [5] gave a description of all the irreducible complex characters of the general linear group $GL_n(k)$ with coefficients in a finite field k. In particular, he proved an interesting formula for the inner product of two class functions on $GL_n(k)$. This formula is very suggestive and may be regarded as an analogue of Weyl's integration formula used in the character theory of compact semisimple Lie groups. However, there exists a disadvantage in the Green's method. Namely, it is too combinatorial theoretic and there seems no direct way of relating it to the structure of GL_n as a linear algebraic group defined over k. The purpose of this paper is to prove a general inner product formula for certain type of class functions on a finite Chevalley group $G(k)$ and to show, when $G=GL_n$, that this provides a new interpretation and proof of the Green's formula.

The main contents of the paper are as follows. In section 1 and 2 we recall some known results on class functions on finite groups and reductive linear algebraic groups. In section 3 we prove the main theorem (Theorem 3.1) mentioned above. Section 4 is devoted to prove a key lemma (Lemma 3.3). In the proof, the following theorem due to R. Steinberg [13] plays an important role:

Let G be a connected reductive linear algebraic group defined over k. Then the number of maximal tori of G defined over k equals to that of unipotent elements of G defined over k.

In sectoin 5 we consider the special case $G=GL_n$ and show that the inner product formula of Green follows easily from Theorem 3.1.

It can be conjectured (see, for example, [6] [7] [8]) that a similar formula exists for general G. Our main theorem in section 3 may be considered as a first step in this direction.

Notations

For a group G and a subset X of G, $Z_G(X)$, $N_G(X)$ denote the centralizer and normalizer of X in G and $C_G(X)$ the G-conjugacy class of X. If G is a linear algebraic group defined over a finite field k, $G(k)$ denotes the finite group of its k-rational elements and G_0 the identity component of G. $Z_{G(k)}(X)$, $N_{G(k)}(X)$ de-