ON THE PATHWISE UNIQUENESS OF
SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC
DIFFERENTIAL EQUATIONS

SHINTARO NAKAO

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Introduction

In this paper, we shall discuss a problem of the pathwise uniqueness for solutions of one-dimensional stochastic differential equations. Let \( a(x) \) and \( b(x) \) be bounded Borel measurable functions defined on \( \mathbb{R} \). We shall consider the following one-dimensional It\( \dot{o} \)'s stochastic differential equation;

\[
\frac{dx_t}{dt} = a(x_t)dB_t + b(x_t)dt.
\]

K. It\( \dot{o} \) [1] proved that, if \( a(x) \) and \( b(x) \) are Lipschitz continuous, a solution is unique and it can be constructed on a given Brownian motion \( B_t \). On the other hand, if \( |a(x)| \) is bounded from below by a positive constant (i.e. uniformly positive), then a solution of (1) exists and it is unique in the law sense. This follows easily from a general result of one-dimensional diffusions (cf. [2]). However, though the distribution of \( \{x_s, B_s\} \) is unique, \( x_t \) is not always expressed as a measurable function of \( x_0 \) and \( \{B_s, s \leq t\} \). For example, if \( a(x) = \text{sgn} \, x, \ a(0) = 1 \) and \( x_0 \equiv 0 \), it is not difficult to see that \( \sigma \{\{x_s; s \leq t\} = \sigma \{B_s; s \leq t\} \).

Here, we will show that, if \( a(x) \) is uniformly positive and of bounded variation on any compact interval, then the pathwise uniqueness holds for (1). This implies, in particular, that \( x_t \) is expressed as a measurable function of \( x_0 \) and \( \{B_s, s \leq t\} \) (cf. [5]). In this direction, M. Motoo (unpublished) already proved that the pathwise uniqueness holds for (1) if \( a(x) \) is uniformly positive and Lipschitz continuous and if \( b(x) \) is bounded measurable. Also, T. Yamada and S. Watanabe [5] proved the pathwise uniqueness of (1) if \( a(x) \) is Holder continuous of exponent \( \frac{1}{2} \) and \( b(x) \) is Lipschitz continuous. Our above mentioned result may be interesting in a point that it applies for many discontinuous \( a(x) \). It is still an open question whether only the uniform positivity of \( a(x) \) implies the pathwise uniqueness.

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