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DISCONTINUOUS SUBGROUPS OF EXTENSIONS OF SEMI-SIMPLE LIE GROUPS

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Introduction

Let G be a semi-simple Lie group acting as a group of linear transformations on the vector space V, K a maximal compact subgroup of G, Γ a discrete subgroup of normalizing a lattice L in V and such that $\Gamma\backslash G$ has finite invariant measure. In this paper we investigate deformations of and cohomology groups attached to $\Gamma\cdot L\subset G\cdot V$, where $\cdot$ donotes semi-direct product. In fact, we give a local description of the space of homomorphisms of $\Gamma\cdot L$ into $G\cdot V$ topologized by compact-open topology, and compute $H^1(\Gamma.L, Ad)$ in certain cases. We also introduce the notion of $\Gamma\cdot L$-invariant form à la Matsushima-Murakami and prove a type decomposition theorem for harmonic forms on $G\cdot V/K$. A special case of this theorem was first proved by Kuga [6].

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1. Deformations of $\Gamma\cdot L$

Let H be a Lie group and Λ discrete subgroup of H. By a *deformation* of Λ in H we mean a family r_t (t ranging over an open interval containing 0) of homomorphisms of Λ into H depending in a C^∞ fashion on t and such that r_0 is the canonical injection of Λ into H. Let $R(\lambda)$ be the tangent vector to $r_t(\lambda)$ at $t=0$. Then $Z\colon \lambda\to R(\lambda)\lambda^{-1}$ is a crossed homomorphism of Λ into the Lie algebra $\mathfrak{H}$ of H for the adjoint action of Λ on $\mathfrak{H}$. ($\mathfrak{H}$ is identified with the tangent space to H at e.) In fact, differentiating the relation $r_t(\lambda\cdot\lambda')=r_t(\lambda)r_t(\lambda')$ we obtain $R(\lambda\cdot\lambda')=R(\lambda)\lambda'+\lambda R(\lambda')$. Upon right multiplication by $\lambda'^{-1}\lambda^{-1}$ we get:

$$R(\lambda\cdot\lambda')\lambda'^{-1}\lambda^{-1} = R(\lambda)\lambda^{-1}+\lambda(R(\lambda')\lambda'^{-1})\lambda^{-1}$$

Since $R(\lambda)\lambda^{-1}\in\mathfrak{H}$ we have proved the assertion. The mapping $Z\colon \lambda\to R(\lambda)\lambda^{-1}$ is called the *crossed homomorphism tangent to* the deformation r_t of Λ.

Two deformations r_t and s_t of Λ are *equivalent* if there is a smooth curve h_t in H such that $r_t(\lambda)=h_t s_t(\lambda)h_t^{-1}$ for all $\lambda\in\Lambda$ and sufficiently small t.