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ON PRIME IDEALS OF A WITT RING OVER A LOCAL RING

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In this note, we consider commutative local rings with invertible element 2, and give a relation between an ordered local ring and a prime ideal of Witt ring over it which is a generalization of the results of Lorenz and Leicht [3] related to prime ideals of Witt ring over a field. By [5], any non-degenerate and finitely generated projective quadratic module (V, q) over a local ring R can be written as a form $(V, q)=\langle a_1\rangle\perp\langle a_2\rangle\perp\cdots\perp\langle a_r\rangle$, where a_i is in the unit group $U(R)$ of R and $\langle a_i\rangle$ denotes a rank one free quadratic submodule $(Rv_i, q|Rv_i)$ such that $q(v_i)=\frac{a_i}{2}$. If, for any element a in $U(R)$, the element having the representative $\langle a\rangle$ in the Witt ring $W(R)$ is denoted by a, then any element of $W(R)$ can be written as a sum of elements of $U(R)$. We use $\perp$, $\top$ and $\otimes$ for the notations of sum, difference and product in $W(R)$. In §1, we have essentially same argument for Witt ring over a local ring as one in [3]. In §2, we study about an ordered local ring R which is an ordered ring such that every unit in R is either >0 or <0, and give a generalization of Sylvester's theorem. In §3, we give an one to one correspondence between such orderings on R and prime ideals $\mathfrak{P}$ of $W(R)$ such that $W(R)/\mathfrak{P}\approx Z$. Throughout this paper, we assume that the ring R is commutative local ring with invertible element 2, and every R-module is unitary.

1. Let R be a local ring with the maximal ideal $\mathfrak{m}$ and the unit group $U(R)$. Since $\langle a\rangle\otimes_R\langle b\rangle\approx\langle ab\rangle$ for $a, b\in U(R)$, we have $(a\perp 1)\otimes(a\top 1)=a^2\top 1=1\perp(-1)=0$ in $W(R)$ for any a in $U(R)$. Therefore, we have the following analogous argument on local ring R to [3]. If $\mathfrak{P}$ is any prime ideal of $W(R)$, then any element a in $U(R)$ is either $a\equiv 1 \pmod{\mathfrak{P}}$ or $a\equiv -1 \pmod{\mathfrak{P}}$. We denote $\varepsilon_{\mathfrak{P}}(a)=1$ or -1, if $a\equiv 1 \pmod{\mathfrak{P}}$ or $a\equiv -1 \pmod{\mathfrak{P}}$, respectively. Then for any element $\alpha\in W(R)$, say $\alpha=a_1\perp a_2\perp\cdots\perp a_n$ for $a_i\in U(R)$, we have $\alpha\equiv\varepsilon_{\mathfrak{P}}(a_1)\perp\varepsilon_{\mathfrak{P}}(a_2)\perp\cdots\perp\varepsilon_{\mathfrak{P}}(a_n) \pmod{\mathfrak{P}}$, therefore there exists an epimorphism $Z\rightarrow W(R)/\mathfrak{P}$, and so $W(R)/\mathfrak{P}\approx Z$ or $\approx Z/(p)$ for some prime number p in the integers Z. Accordingly, we have