Yagi, K.
Osaka J. Math.
7 (1970), 457-475

ON COMPACT HOMOGENEOUS AFFINE MANIFOLDS

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(Received June 9, 1970)

Introduction

If a differentiable manifold M is provided with an affine connection whose torsion and curvature vanish identically, we call M an affine manifold. The study of affine manifolds has been the subject of a number of recent publications including the papers by Auslander, Charlap, Koszul, Kamber and Tondeur, and Wolf. A general reference for the study of affine manifolds is [3], [4] or [6]. The subject of this paper is to study homogeneous affine manifolds.

If, for an affine manifold M, **aut** (M) denotes the Lie algebra of all infinitesimal affine transformations, then **aut** (M) has an associative algebra structure satisfying 1) $X\cdot Y-Y\cdot X=[X, Y]$ and 2) the isotropy subalgebra $\mathbf{aut}(M)_p=\{X\in\mathbf{aut}(M)\,|\,X_p=0\}$ at $p\in M$ is a left ideal of **aut** (M) (Theorem 1.2). Our study is essentially based upon these properties of **aut** (M). A pair $(\mathfrak{g}, \mathfrak{a})$ of a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is called an $\mathcal{A}$-pair if $\mathfrak{g}$ has an associative algebra structure satisfying the above 1) and 2) for the subalgebra $\mathfrak{a}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and A a closed subgroup of G with Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Then if $(\mathfrak{g}, \mathfrak{a})$ is an $\mathcal{A}$-pair, then the homogeneous space G/A has a unique G-invariant flat affine connection ∇ satisfying $\nabla_{X^*}Y^*=(Y\cdot X)^*$ where X^* denotes the vector field on G/A induced by the action of $\exp tX$ (Theorem 2.2). We call such a homogeneous affine manifold an $\mathcal{A}$-space. Then a compact homogeneous affine manifold turns out to be an $\mathcal{A}$-space (Theorem 2.4).

To each $\mathcal{A}$-pair $(\mathfrak{g}, \mathfrak{a})$, we can associate in a canonical way a pair (G, A) of Lie groups such that the Lie algebras of G and A are $\mathfrak{g}$ and $\mathfrak{a}$ respectively, and A is a closed subgroup of G (§4). Then for such a pair (G, A) of a Lie group and a closed subgroup, the $\mathcal{A}$-space G/A is embedded equivariantly into an affine space as a domain, which is called an $\mathcal{A}$-domain (Theorem 4.5).

The F-Stiefel manifold $V_r(F^n)$, consisting of all r linear frames in F^n ($F=R$, C or H), is naturally imbedded into the affine space F^{nr} as a domain,

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This work was presented to the Graduate School of the University of Notre Dame in partial fulfillment of the requirement for the Ph.D. degree.