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ON UNRAMIFIED GALOIS EXTENSIONS OF QUADRATIC NUMBER FIELDS

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Introduction

Let n be a given natural number greater than 1. It was shown by several authors that there exist infinitely many imaginary quadratic number fields whose ideal class numbers are multiples of n (Nagel [9], Humbert [7], Ankeny and Chowla [1], Kuroda [8]). For real quadratic number fields, however, there seems to be no corresponding result except special cases $n=2^i$ $(i=1, 2, \cdots)$ (Gauss [4]) and $n=3$ (Honda [6]).

In part I of this paper we show the infiniteness of the number of such real quadratic number fields for every natural number n (Corollary 1 of Theorem 2), by modifying the method used in [9]. At the same time we get an infinite number of imaginary quadratic number fields F each of which has a subgroup of order n^2 isomorphic to the direct product of two cyclic groups of order n in its ideal class group. Moreover we can impose certain conditions on the behaviour of finite number of primes in F. Our method is sketched as follows: In the first place, we construct a quadratic number field which has two ideal classes $\mathfrak{a}$, $\mathfrak{a}'$ and satisfies some local conditions on its discriminant D. In case $D<0$, both of them are of order n and independent. In case $D>0$, neither of them may be of order n because of the existence of non-trivial units but the subgroup $\langle\mathfrak{a}, \mathfrak{a}'\rangle$ generated by them contains at least an ideal class $\mathfrak{b}$ of order n. Next we show that such fields exist infinitely for either case, using the local conditions on D.

According to the class field theory, the ideal class group of a number field is closely related to the maximal unramified abelian extension of the field. In part II we study other types of unramified Galois extensions of quadratic number fields. First, as a special case of Hibert's irreducibility theorm, we construct (infinitely many) Galois extensions of the rational number field $\boldsymbol{Q}$ whose Galois groups are isomorphic to the symmetric group S_n of degree n and each of which is defined as a minimal splitting field of a trinomial equation

$$X^n+aX+b=0 \qquad (a, b\in \boldsymbol{Z}).$$

These fields are unramified over the quadratic number fields corresponding to