Tsushima, Y.
Osaka J. Math.
5 (1968), 89-98

A GROUP ALGEBRA OF A p-SOLVABLE GROUP

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(Received February 2, 1968)

1. Introduction

This paper is a sequel to our earlier one [6] and we are concerned also with the radical of a group algebra of a finite group, especially of a p-solvable group. Let G be a finite group of order $|G|=p^n g'$, where p is a fixed prime number, n is an integer$\geqq 0$ and $(p, g')=1$. Let S_p be a Sylow p-group of G and k a field of characteristic p. We denote by $\mathfrak{N}$ the radical of the group algebra kG (These notations will be fixed throughout this paper). Let B be a block of defect d in kG. Then $\mathfrak{N}B$ is the radical of B. First we shall show $(\mathfrak{N}B)^{p^d}=0$, when G is solvable or a p-solvable group with an abelian Sylow p-group. In §3, we assume S_p is abelian. Let H be a normal subgroup of G and $\mathfrak{R}$ the radical of kH. It follows from Clifford's Theorem that $\mathfrak{R}\subset\mathfrak{N}$, hence $\mathfrak{L}=kG\cdot\mathfrak{R}=\mathfrak{R}\cdot kG$ is a two sided ideal contained in $\mathfrak{N}$. If $[G:H]$ is prime to p, we have $\mathfrak{L}=\mathfrak{N}$ (Proposition 1 [6]). In another extreme, suppose $[G:H]=p$. Then we can show there exists a central element c in $\mathfrak{N}$ such that $\mathfrak{N}=\mathfrak{L}+(kG)c$. Hence if G is p-solvable, $\mathfrak{N}$ can be constructed somewhat explicitly using a special type of a normal sequence of G (Theorem 2). If S_p is normal in G, then $\mathfrak{N}$ is generated over kG by the radical of kS_p ([7] or Proposition 1 [6]). Hence Theorem 2 may be considered as a generalization of the above fact to the case that S_p is abelian. In the special case that S_p is cyclic, our main results will be improved in the final section.

Besides the notation introduced above we use the following; H will always denote a normal subgroup of G, $\mathfrak{R}$ the radical of kH and $\mathfrak{L}=kG\cdot\mathfrak{R}$. For a subset T in G, $N_G(T)$ and $C_G(T)$ are the normalizer and the centralizer of T in G. For an element x in G, $[x]$ denotes the sum of the elements in the conjugate class contaning x. Finally, we assume k is a splitting field for every subgroup of G.

2. Radical of a block

We begin with some considerations on the central idempotents. Let $\mathfrak{A}=\{\eta_i\}$ be the set of the block idempotents in kH. G induces a permutation group on $\mathfrak{A}$ by $\eta_i\to g^{-1}\eta_i g$, $g\in G$. Let $\tilde{\mathfrak{F}}_1\cdots\tilde{\mathfrak{F}}_s$, be the set of transitivity. We use the