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THE DEDEKIND DIFFERENT AND THE HOMOLOGICAL DIFFERENT

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We intend, in this paper, to define the Dedekind different of an algebra over a commutative ring and to study the properties of this different. S. Endo and the author [4] defined the reduced trace of a central separable algebra over a commutative ring. Using this reduced trace, we can define as usual the Dedekind different of an algebra. Let R be a commutative ring and A be an R-algebra which is a finitely generated projective R-module. We assume that $\mathfrak{A}=K\otimes A$ is a central separable K-algebra, where K is the total quotient ring of R. Let t be the reduced trace of $\mathfrak{A}$. The two sided A-submodule $C=\{x\in\mathfrak{A}\mid t(xA)\subset R\}$ of $\mathfrak{A}$ is called the *complementary module* of A, and $D=\{x\in\mathfrak{A}\mid xC\subset A\}$ is called the *Dedekind different* of A. D is also a two sided A-module. We shall first show that the reduced trace induces an epimorphism of the Dedekind different to the homological different which was defined in [2]. This fact was shown by Fossum in the case that R is an integrally closed Noetherian domain ([5]). Secondly we shall give a complete generalization of DeMeyer's theorem ([3], Theorem 4), and finally we shall give a generalization of the "different theorem" on maximal orders over Dedekind domains in central simple algebras. Throughout this note we assume that rings have unit elements, that modules are unitary and that algebras are finitely generated as modules.

1. Let R be a commutative ring, A be an R-algebra and A^e be the enveloping algebra of A. $J(A)$ (or briefly, J) denotes the kernel of the canonical A^e-epimorphism φ: $A^e\to A$ given by $\varphi(x\otimes y^0)=xy$, and $N(A)$ (or briefly, N) denotes the right annihilator of $J(A)$ in A^e.

Lemma 1. *Let A be a full matrix algebra of degree n over R. N is an R-free submodule of A^e with basis $\sum_{i=1}^{n} e_{ij}\otimes e_{ki}^0$, $1\leqq j, k\leqq n$, where e_{ij} denotes the (i, j)-matrix unit.*

Proof. Let $\alpha=\sum_{ijkl} a_{ijkl}(e_{ij}\otimes e_{kl}^o)$ be in N. Since $e_{rr}\otimes 1^0-1\otimes e_{rr}^o$ is in J, it is annihilated by α, so we get $\sum_{jkl} a_{rjkl}(e_{rj}\otimes e_{kl}^o)=\sum_{ijk} a_{ijkr}(e_{ij}\otimes e_{kr}^o)$. Hence, $i\neq l$ implies $a_{ijkl}=0$. So, α is expressed as $\alpha=\sum_{ijk} a_{ijk}(e_{ij}\otimes e_{ki}^o)$. Again, by