Harada, M.
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QF-3 AND SEMI-PRIMARY PP-RINGS I

MANABU HARADA

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Recently the author has given a characterization of semi-primary hereditary ring in [4]. Furthermore, those results in [4] have been extended to a semi-primary PP-ring in [3], (a ring A is called a *left PP-ring if every principal left ideal in A is A-projective*).

This short note is a continuous work of [3] and [4]. Let K be a field and A an algebra over K with finite dimension. A is called a *QF-3 algebra if A has a unique minimal faithful representation* ([10]). Mochizuki has considered a hereditary QF-3 algebra in [6].

In this note we shall study a PP-ring with minimal condition or of semi-primary. To this purpose we generalize a notion of QF-3 algebra in a case of ring. We call *A left (resp. right) QF-3 ring if A has a faithful, injective, projective left (resp. right) ideal*, (cf. [5], Theorems 3.1 and 3.2).

Let $1=\sum E_i$ be a decomposition of the identity element 1 of a semi-primary ring A into a sum of mutually orthogonal idempotents such that E_i modulo the radical N is the identity element of simple component of A/N. *If Ax is A-projective for all $x\in E_iAE_j$*, we call A a *partially PP-ring*, (see [3], §2). Such a class of rings contains properly classes of semi-primary hereditary rings and PP-rings.

Our main theorems are as follows: *Let A be directly indecomposable and a left QF-3 ring and semi-primary partially PP-ring. Then 1) there exists a unique primitive idempotent e in A (up to isomorphism) such that $eN=(0)$ and every indecomposable left injective ideal in A is faithful, projective and isomorphic to Ae. Furthermore, A is a right QF-3 ring. 2) Let $B=\mathrm{Hom}_{eAe}(Ae, Ae)$, where Ae is regarded as a right eAe-module. Then eAe is a division ring and $B=(eAe)_n$*[1]*. B is a left and right injective envelope of A as an A-module and B is A-projective. Furthermore, if A is hereditary, then A is a generalized uniserial ring whose basic ring is of triangular matrices over a division ring.* (Mochizuki proved in [6] the above fact 2) in a case of hereditary algebra over a field with finite dimension).

1) $(A)_n$ means a ring of matrices over a ring A with degree n.