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CONGRUENCE RELATIONS AND CONGRUENCE CLASSES IN LATTICES

Dedicated to Professor K. Shoda on his sixtieth birthday

By

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1. Introduction

The general theory of abstract algebraic systems (algebras) introduced by Professor Shoda [7] has been successful not only to unify earlier results about many algebraic systems (groups, rings, lattices, etc.) but to develop further investigations into each individual system. The present paper is an additional work in those lines. We shall first consider the relation between congruence relations and congruence classes on universal algebras and next inquire precisely into the same problem on lattices.

In the present paper by an *algebra* A we shall mean, following Birkhoff [1], [2], a system with a number of operations f_λ: $(x_1, \cdots, x_n) \in A\times \cdots \times A \rightarrow f_\lambda(x_1, \cdots, x_n) \in A$. A homomorphism θ of A onto an algebra $B=\theta(A)$ yields a congruence relation on A, which shall be written $x\equiv y(\theta)$ or $x\theta y$; so

$$x\theta y \rightleftarrows \theta(x) = \theta(y)\,.$$

Conversely a congruence relation θ on A yields a homomorphism of A onto the algebra $\theta(A)$ of classes $S(a, \theta)=\{x\,;\, x\theta a\}$, which we shall denote also by the same notation θ.

For the investigation into the structure of algebras such as groups, rings, etc., the following properties on the congruence relations work effectively:

(α) Every congruence relation is determined by the congruence class containing a fixed element a; namely

$$S(a, \theta) = S(a, \varphi) \quad \text{implies} \quad \theta = \varphi\,,$$

(β) Congruence relations on A are permutable.
Some algebras however do not necessarily possess those properties. In this respect Birkhoff [2] has proposed the following problems.