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REMARKS ON PRINCIPAL IDEAL RINGS

To Prof. K. Shoda

By

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The study of non commutative (associative) principal ideal rings and its unique factorization has been carried out extensively by Asano [1] and others and was almost completely summarized by Jacobson in [3] (Ch. 3). The motivation to this study was the factorization of differential polynomials which constitute a non-commutative principal ideal ring (Ore [4]) and the factorization of elements in algebras, in particular in matrices with integral elements, or in general commutative principal ideal rings (e.g. Dickson, "Algebras and their arithmetics", Chicago 1923). The latter is closely connected with the subject of elementary divisors and invariant factors.

The present paper contains two additional notes to this subject and a generalization of a notion of local rings, which appears in arithmetical considerations of commutative rings. The first section deals with modules over locally principal ideal rings and applies the result to obtain some consequences in the direction of Hermite-ring which are stronger than obtained in [2]. In the second section, it is shown that the unique factorization theorem, in the sense of [4] or [5] (Ch. 3) holds in some more general principal ideal rings and this is applied to obtain a straightforward proof of Nakayama's theorem ([5]) on the uniqueness of the elementary divisors of matrices over principal ideal rings. The last part deals with a generalized notion of local rings of principal ideals ring at a prime p, and it is shown that such a ring exists if one takes into consideration all the primes equivalent to p.

1. Matrices and modules over principal ideal rings

Let R be an associative (not necessarily commutative) ring with a unit 1, and R_n be the ring of all $n \times n$ matrices over R.

We shall deal with rings R satisfying the following:

(R1) Every *finitely generated* right ideal is a principal right ideal.