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ON FIXED POINT FREE INVOLUTIONS OF $S^1 \times S^2$

Dedicated to Professor K. Shoda on his sixtieth birthday

By

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Introduction

In 1958, J. H. C. Whitehead [10] generalized the sphere theorem by C. D. Papakyriakopoulos in the following way:

WHITEHEAD'S SPHERE THEOREM. *Let M be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_2(M) \neq 0$. Then there exists a 2-sphere S semi-linearly embedded in M, such that $S \neq 0$[1] in M.*

As the example $S^1 \times P^2$[2] (S^k means k-sphere) shows, the above sphere theorem does not hold generally for non-orientable 3-manifolds. Therefore it remains as a question that for what 3-manifolds the sphere theorem does not hold? This problem naturally leads to the fixed point free involution (homeomorphism on itself of order 2) of $S^1 \times S^2$ as Theorem 2 of § 3 in this paper shows.

The main purpose of this paper is to prove the following

Theorem 1. *If T is a fixed point free involution of $S^1 \times S^2$, and if M is the 3-manifold obtained by identifying x and Tx in $S^1 \times S^2$, then M is either homeomorphic to* (1) $S^1 \times S^2$, *or* (2) *3-dimensional Klein Bottle*[3] (*we denote it by* K^3), *or* (3) $S^1 \times P^2$, *or* (4) $P^3 \#$[4] P^3.

This theorem may be regarded as an analogy of the following

Theorem (G. R. LIVESAY [4]). *If T is a fixed point free involution*

1) $\neq 0$ means not homotopic to a constant.

2) P^2 is the real projective plane.

3) 3-dimensional Klein Bottle is defined as follows: let S_0, S_1 be the boundaries of $S^2 \times [0, 1]$. Then S_0 S_1 have the orientations induced from the orientation of $S^2 \times [0, 1]$. Let f be an orientation preseving homeorphirm from S_0 to S_1. Identifying S_0 with S_1 by f in $S^2 \times [0. 1]$, we obtain a non-orientable closed 3-manifold which we call 3-dim. Klein Bottle.

4) P^3 is the projective space. $P^3 \# P^3$ is defined as follows: Let E', E'' be two open 3-cells in P^3, P^3 respectively. Matching the boundaries of $P^3 - E'$ and $P^3 - E''$, we obtain a new closed 3-manifold which we denote $P^3 \# P^3$.