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Note on Axiomatic Set Theory I. The Independence of Zermelo's "Aussonderungsaxiom" from Other Axioms of Set Theory

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In this paper axioms of set theory mean Gödel's axioms of set theory in [2], which are modified so that the axiom C 2 postulates directly the existence of all elements of a given set and C 3 the existence of power set.

In what follows we prove that Zermelo's "Aussonderungsaxiom" is independent of other axioms of set theory. A fortiori, the independence of Gödel's axiom C 4 (Fraenkel's "axiom of substitution"), is proved since the axiom of substitution implies the "Aussonderungsaxiom".

The proof is carried out by constructing an inner model Λ for the axioms A, B, C 1, C 2, C 3, D and E under the axioms A, B, C 1, C 2 and C 4, which does not satisfy the "Aussonderungsaxiom". The idea appears already in [1]. However the proof in [1] is not formal.

In § 1, we give the axioms of set theory.

In § 2, we construct an inner model Λ under the axioms A, B, C 1, C 2 and C 4. Here we notice that the axioms do not imply the existence of power set of any set.

In § 3, we prove some preliminary results with respect to the model Λ.

In § 4, we prove that the model Λ satisfies the axioms A, B, C 1, 2, 3, D and E.

In § 5, we prove that the model Λ does not satisfy the "Aussonderungsaxiom".

§ 1. The axiom system of set theory

In what follows we apply Gödel's notations in [2] in most cases. For logical notations we use following symbols, $\vee$ (or), $\cdot$ (and), $\sim$ (not), $\supset$ (implies), $\equiv$ (equivalence), $=$ (identity), $(\exists X)$ (there is an X), (X) (for all X) and $(\exists! X)$ (there is exactly one X). The system has three primitive notions: class, denoted by $\mathfrak{Cls}$; set, denoted by $\mathfrak{M}$; and the diadic relation $\in$ between class and class, class and set, set and class, or