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Global Stability Criteria for Differential Systems

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Consider the real differential system

$$\mathcal{S})\qquad \frac{dx^i}{dt} = f^i(x^1, \cdots, x^n) \qquad i = 1, 2, \cdots, n$$

with the real vector-valued function $f(x)$ in class C^1 in the real vector space R^n. The local stability theorem of A. Liapounov [8, and 2, P. 341] states that if the origin is a critical point,

$$f(0) = 0 ,$$

and if the eigenvalues of the Jacobian matrix $J(0)$, where

$$J^i_j(x) = \frac{\partial f^i}{\partial x^j}(x) ,$$

have negative real parts, then each solution of $\mathcal{S})$ which initiates near the origin must approach the origin $t \to +\infty$. We shall extend this result to a global stability criterion which generalizes a theorem of N. N. Krasovski [3, 4].

For the differential system $\mathcal{S})$ consider the local eigenvalues $\lambda_1(x^1, \cdots, x^n), \cdots, \lambda_n(x^1, \cdots, x^n)$ which are the roots of the characteristic polynomial

$$|J(x) - \lambda I| .$$

If

$$\operatorname{Re} \lambda_i(x^1, \cdots, x^n) < 0 \qquad i = 1, 2, \cdots, n$$

everywhere in R^n, then it has been conjectured [1] that each solution curve of $\mathcal{S})$ must approach a critical point of $\mathcal{S})$ as $t \to +\infty$. This is clearly true for $n=1$ (when $\mathcal{S})$ has a critical point) but it has been established in only particular cases when $n \geq 2$.

Note that an affine coordinate change in R^n,

$$y^i = A^i_j x^j + a^i , \qquad \det A^i_j \neq 0 ,$$

transforms the system $\mathcal{S})$ to