Murasugi, Kunio
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On Alternating Knots

By Kunio MURASUGI

Introduction

Let k be a knot[1] in 3-sphere S^3. Let k be an image of the regular projection of k onto $S^2 \subset S^3$. A knot projection K is said to be *alternating* if and only if it is connected and, as one follows along this knot, undercrossings and overcrossings alternate[2]. A knot is said to be *alternating* if it prossesses an alternating projection. There will be, of course, non-alternating knots. In fact, the first proof of their existence was given by Bankwitz in 1930 [3]. We do not know the general method by which we can decide whether or not a given knot projection represents an alternating knot. But some good methods have been found up to present. Recently, R. H. Crowell proved the theorem (cf. Theorem (6. 5) [5]) which much improved the Bankwitz's theorem (cf. Satz, p. 145 [3]). He showed by means of this theorem that seven of eleven non-alternating projections, in the Knot Table at the end of Reidemeister's Knotentheorie [11], represent non-alternating knots.

In a previous paper [9] we gave a necessary condition for a given knot to be alternating by means of its Alexander polynomial (cf. [4], [8]). In the present paper, in order to characterize the Alexander polynomial of the alternating knot, we shall assign a matrix, called the *knot matrix*, to an alternating knot. The relation between the knot matrix and its Alexander polynomial is expressed in Theorem 1. 17 which is the fundamental theorem of the present paper. From this theorem simply follow the main theorems in [4], [7], [8], [9] (Theorem 3. 8, Theorem 3. 13). Theorem 3. 12 is a simple application of Theorem 1. 17 and it plays a particular rôle in § 3.

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1) A *knot* is an oriented polygonal simple closed curve. A *link* of multiplicity μ is the union of μ ordered, pairwise disjoint knots. In the present paper we do not distinct exactly between links and knots, except the cases 3.7 and 3.8. Thus, by a *knot* (of multiplicity μ) is meant an ordinary knot or link according to $\mu=1$ or $\mu>1$.

2) For any knot we may select a "point at infinity" $\infty \in S^3-k$ and consider a Cartesian coordinate system $R\times R\times R=S^3-\infty$. The projection $p: S^3\rightarrow S^2$ is defined by $p(\infty)=\infty$ and $p(x, y, z)=(x, y)$. For each double point $p(a)=p(b)$, one of a and b with the larger z-coordinate is called the *overcrossing* and the other the *undercrossing*.