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Decompositions of a Completely Simple Semigroup

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§ 1. Introduction.

In this paper we shall study the method of finding all the decompositions of a completely simple semigroup and shall apply the result to the two special cases: an indecomposable completely simple semigroup [2] and a $\mathfrak{H}$-semigroup [4]. By a decomposition of a semigroup S we mean a classification of the elements of S due to a congruence relation in S. Let S be a completely simple semigroup throughout this paper. According to Rees [1], it is faithfully represented as a regular matrix semigroup whose ground group is G and whose defining matrix semigroup is $P=(p_{\mu\lambda})$, $\mu\in L$, $\lambda\in M$, that is, either $S=\{(x\,;\,\lambda\mu)\,|\,x\in G,\ \mu\in L,\ \lambda\in M\}$ or S with a two-sided zero 0, where the multiplication is defined as

$$(x\,;\,\lambda\mu)(y\,;\,\xi\eta)=\begin{cases}(xp_{\mu\xi}y\,;\,\lambda\eta) & \text{if } p_{\mu\xi}\neq 0\\ 0 & \text{if } p_{\mu\xi}=0 \text{ and hence } S \text{ has } 0.\end{cases}$$

For the sake of simplicity S is denoted as

$$Simp.(G,\ 0\,;\ P)\quad\text{or}\quad Simp.(G\,;\ P)$$

according as S has 0 or not. L and M may be considered as a right-singular semigroup and a left-singular semigroup respectively [5].

§ 2. Normal Form of Defining Matrix.

We define two equivalence relations $\overset{0}{\underset{M}{\sim}}$ and $\overset{0}{\underset{L}{\sim}}$ in M and L respectively: we mean by $\lambda\overset{0}{\underset{M}{\sim}}\sigma$ that $p_{\eta\lambda}\neq 0$ if and only if $p_{\eta\sigma}\neq 0$ for all $\eta\in L$; by $\mu\overset{0}{\underset{L}{\sim}}\tau$ that $p_{\mu\xi}\neq 0$ if and only if $p_{\tau\xi}\neq 0$ for all $\xi\in M$. Let $L=\sum_{\mathfrak{l}} L_{\mathfrak{l}}$, and $M=\sum_{\mathfrak{m}} M_{\mathfrak{m}}$ be the classifications of the elements of L and M due to the relations $\overset{0}{\underset{L}{\sim}}$ and $\overset{0}{\underset{M}{\sim}}$ respectively.

Lemma 1. *A defining matrix is equivalent to one which satisfies the following two conditions. Let e be a unit of G.*

(1) *For any $\mathfrak{m}$, there is $\alpha(\mathfrak{m})\in L$ such that $p_{\alpha(\mathfrak{m}),\xi}=e$ for all $\xi\in M_{\mathfrak{m}}$.*

(2) *For any $\mathfrak{l}$, there is $\beta(\mathfrak{l})\in M$ such that $p_{\eta,\beta(\mathfrak{l})}=e$ for all $\eta\in L_{\mathfrak{l}}$.*