Yamabe, Hidehiko
Osaka Math. J.
12 (1960), 21-37.

On a Deformation of Riemannian Structures on Compact Manifolds

By Hidehiko YAMABE

1. The purpose of this paper is to prove that every compact C^∞-Riemannian manifold with at least 3 dimensions can be deformed conformally to a C^∞-Riemannian structure of constant scalar curvature.

Let S be a d-dimensional C^∞-Riemannian manifold with $d\geqq 3$, and denote its fundamental positive definite tensor by g_{ij}. Throughout this paper we will use the definitions and notations of the book "Curvature and Betti numbers" by K. Yano and S. Bochner, unless otherwise stated. The volume element is written as dV. The total volume is assumed to be 1.

Here we are going to present the outline of the proof. Consider a conformal transformation of a Riemannian structure

$$\bar{g}_{ij} = e^{2\rho} g_{ij}. \tag{1.1}$$

Then the connection coefficients $\bar{\Gamma}^i_{jk}$ corresponding to $\bar{g}_{ij}$ are expressed as[1)]

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \rho_k \delta^i_j + \rho_j \delta^i_k - \rho^i g_{jk}, \tag{1.2}$$

where

$$\rho_i = \frac{\partial \rho}{\partial x^i}. \tag{1.3}$$

From (1.2)

$$\bar{R}^i_{jkl} = R^i_{jkl} - \rho_{jk}\delta^i_l + \rho_{jl}\delta^i_k - g_{jk}\rho^i_l + g_{jl}\rho^i_k \tag{1.4}$$

where

$$\rho_{jk} = \rho_{j,k} - \rho_j \rho_k + \frac{1}{2} g^{\alpha\beta}\rho_\alpha \rho_\beta g_{jk}. \tag{1.5}$$

Hence

$$\bar{R}_{jk} = R_{jk} - (d-2)\rho_{jk} - g_{jk}\rho^\alpha_\alpha \tag{1.6}$$

and

1) see [5] page 78.