Strict Convexity and Smoothness of Normed Spaces

Junzo WADA

V. L. Klee [11] and M. M. Day [6] have considered various problems on strict convexity and smoothness of normed spaces. In his paper, Day [6] raised several questions. Two of these are the following:

1) Is any $L_\Gamma$-space strictly convexifiable?
2) Is there a nonreflexive nonsparable $scn$ space?

In this paper, we consider these questions. In § 2 we deal with spaces of bounded continuous functions and consider strict convexity and smoothness on these spaces. In § 3 we give a partial answer to the first question, and in § 4 we give an answer to the second, by showing an example of a nonreflexive nonseparable $scn$ space.

§ 1. Preliminary.

Let $E$ be a normed space. If every chord of the unit sphere has its midpoint below the surface of the unit sphere, then $E$ is called strictly convex (written SC); if through every point of the surface of the unit sphere of $E$ there passes a unique hyperplane of support of the unit sphere, then $E$ is called smooth (written SM); if both occur, then $E$ is called SCM. If $E$ is isomorphic to an SM, an SC and an SCM space, then $E$ is called an sm, an sc and an scm space respectively.

If $I$ is an index set, we define:

$m(I)=\text{the space of all bounded real functions on } I \text{ with } ||x||=\text{l.u.b. }_{i \in I} |x(i)|$.

c_o(I)=\text{the subspace of those } x \text{ in } m(I) \text{ for which for each } \varepsilon > 0 \text{ the set of } i \text{ with } |x(i)| \geq \varepsilon \text{ is finite}; \text{ that is, } c_o(I)=\text{the set of functions vanishing at infinity on the discrete space } I.

$l_p(I) \text{ (for } p \geq 1)=\text{the set of those real functions } x \text{ on } I \text{ for which } ||x||_{l_p}=[\sum |x(i)|^p]^{1/p} < +\infty$.

Let $X$ be a topological space. Then $C(X)$ denotes the space of all real-valued bounded continuous functions on $X$ such that the norm $||f||=\sup_{x \in X} |f(x)|$.

1) Numbers in bracket refer to the references cited at the end of the paper.