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Direct, Subdirect Decompositions and Congruence Relations

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1. Introduction

On the direct decompositions of (universal) algebras, applying to all of groups, rings, linear algebras and lattices, many researches have been made, but almost all of them are concerned with the decompositions into a finite number of factors. In the present paper we attempt to extend those earlier results to the case of infinite factors and clarify the structure of some algebras.

By an *algebra* A, we shall mean below a set of elements, together with a number of finitary operations f_α. Each f_α is a single-valued function assigning for some finite $n=n(\alpha)$ to every sequence $(x_1, \cdots, x_n)$ of n elements of A, a value $f_\alpha(x_1, \cdots, x_n)$ in A. A *congruence relation* θ on an algebra A is an equivalence relation $x\equiv y(\theta)$ with the substitution property for each f_α: If $x_i\equiv y_i(\theta)$, then $f_\alpha(x_1, \cdots, x_n)\equiv f_\alpha(y_1, \cdots, y_n)$ (θ). A congruence relation θ on A generates a homomorphism of A onto the algebra $\theta(A)$ of subsets $C(a, \theta)=\{x\,;\, x\equiv a(\theta)\}$ of A, which we denote by the same notation θ. If we define $\theta\leq\varphi$ to mean that $x\equiv y(\theta)$ implies $x\equiv y(\varphi)$, then all congruence relations on A form a complete, upper continuous lattice $\Theta(A)$, which we shall call the *structure lattice* of A. By the (complete) *direct union* $A=\Pi_{\omega\in\Omega}A_\omega$ of algebras A_ω having the same operations f_α is meant the algebra whose elements are the sets $\{x_\omega\,;\, \omega\in\Omega\}$ with $x_\omega\in A_\omega$, in which algebraic combination is performed component by component: If $x^i=\{x^i_\omega\,;\, \omega\in\Omega\}$, then $f_\alpha(x^1, \cdots, x^n)=\{f_\alpha(x^1_\omega, \cdots, x^n_\omega)\,;\, \omega\in\Omega\}$.

Direct factorizations of an algebra A are correlated with the lattice-theoretic properties of congruence relations on A; for instance

THEOREM 1.1. *The representations of an algebra A as a direct union $A=A_1\times\cdots\times A_n$ correspond one-one with the sets of permutable congruence relations $\theta_1, \cdots, \theta_n$ on A satisfying*

$$\theta_1\frown\cdots\frown\theta_n=0 \quad\text{and}\quad (\theta_1\frown\cdots\frown\theta_{i-1})\smile\theta_i=I\ [i=2, \cdots, n].$$

But this theorem does not hold for infinite n. We first intend to obtain the corresponding theorem for infinite n, by introducing the con-