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Algebraic Study of Fundamental Characteristic Classes of Sphere Bundles

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The cohomology theory of principal fibre bundles is extremely developed in these several years by many mathematicians, above all by A. Borel. Owing to this theory, the real cohomology structure of an n-universal bundle for a compact Lie group is completely determined. In particular, its base space, called n-classifying space, has the real cohomology algebra which is isomorphic in dimensions $\leqq n-1$ to the algebra of invariant polynomial functions on the Lie algebra of the structural group. Invariant polynomial functions are those which are invariant under the operations of adjoint group. As for an arbitrary given principal fibre bundle over a cell-complex with a compact Lie group, it is well-known that this bundle is induced by a mapping, say f, from its base space into an n-classifying space. The cohomology homomorphism f^* induced by f maps the cohomology algebra of the n-classifying space into that of the base space and it defines as its image the characteristic algebra whose elements are the so-called characteristic classes of the given bundle. Recall that these notions do not depend on the choices of the n-classifying space and of the mapping f. Referring to the fact mentioned above, as far as we concern real cohomology we may regard the characteristic classes as the images under a certain homomorphism of invariant polynomial functions. As a matter of fact, for a differentiable fibre bundle H. Cartan [2] and S. S. Chern [4] have availed themselves of a homomorphism explicitely given by making use of the curvature form of a connection in the bundle. This homomorphism will serve us as a foundation of the present work.

A sphere bundle is a fibre bundle whose fibre is a sphere acted by a subgroup of the orthogonal group. Its characteristic classes are those of the principal fibre bundle associated to it. As regard to the characteristic classes of a sphere bundle, there are important ones—Pontrjagin classes, Chern classes, etc.—which we call fundamental characteristic classes. These are usually defined as the f^*-images of special cohomology classes in a Grassmann manifold—a concrete n-classifying space for a compact classical group. By detailed studies of Grassmann manifolds