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Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces, II[1]

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The present article is concerned with the equilibrium potential on Riemann surfaces with positive boundary.

1. Let R^* be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. Let $N_n(z, p)$ be a positive function in R_n-R_0 harmonic in R_n-R_0 except one point $p\in R$ such that $N_n(z, p)=0$ on ∂R_0, $\frac{\partial N_n(z, p)}{\partial n}=0$ on ∂R_n and $N_n(z, p)+\log|z-p|$ is harmonic in a neighbourhood of p. Then the $*$–Dirichlet integral of $N_n(z, p)$ taken over R_n-R_0 is $D^*(N_n(z, p))=U_n(p)$, where $U_n(p)=\lim_{z\to p}(N_n(z, p)+\log|z-p|)$ and the $*$–Dirichlet integral is taken with respect to $N_n(z, p)+\log|z-p|$ in the neighbourhood of p. For $N_n(z, p)$ and $N_{n+i}(z, p)$, we have

$$\begin{aligned} &D^*_{R_n-R_0}(N_n(z, p),\ N_{n+i}(z, p)) = D^*_{R_{n+i}-R_0}(N_{n+i}(z, p)) = 2\pi U_{n+i}(p)^{2)}, \\ &D_{R_n-R_0}(N_n(z, p)-N_{n+i}(z, p)) = D^*_{R_n-R_0}(N_n(z, p))-2D^*_{R_n-R_0}(N_n(z, p),\ N_{n+i}(z, p)) \\ &\quad +D^*_{R_n-R_0}(N_{n+i}(z, p)) < D^*_{R_n-R_0}(N_n(z, p))-D^*_{R_{n+i}-R_0}(N_{n+i}(z, p)) \\ &\quad = 2\pi(U_n(p)-U_{n+i}(p)) . \end{aligned}$$

Hence $\{U_n(p)\}$ is decreasing with respect to n. Since $\int_{\partial R_0}\frac{\partial N_n(z, p)}{\partial n}ds=2\pi$ for every n, $\lim_{n=\infty} U_n(p)>-\infty$, whence $\{U_n(p)\}$ converges. Therefore $D_{R_{n+i}-R_0}(N_{n+i}(z, p)-N_n(z, p))$ tends to zero if n and i tend to ∞, which implies that $\{N_n(z, p)\}$ converges in mean. Further $N_n(z, p)=0$ on ∂R_0 yields that $\{N_n(z, p)\}$ converges uniformly to a function $N(z, p)$, which clearly has the minimal $*$–Dirichlet integral over R, in every compact part of R. Clearly by the compactness of ∂R_0, we have $\int_{\partial R_0}\frac{\partial N(z, p)}{\partial n}ds=$

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2) Let $v_r(p)$ be a circular neighbourhood of p with respect to the local parameter: $v_r(p)=E[z\in R: |z-p|<r]$. Then $D^*(N_n(z, p),\ N_{n+i}(z, p))=\int_{\partial v(p)}(N_{n+i}(z, p)+\log|z-p|)\frac{\partial N_n(z, p)}{\partial n}ds$. By letting $r\to 0$, we have $D^*(N_{n+i}(z, p),\ N_n(z, p))=2\pi U_{n+i}(p)$. Clearly $*$-Dirichlet integral reduces to Dirichlet integral when the functions have no pole.