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Mass Distributions on the Ideal Boundaries of Abstract Riemann Surfaces. I[1]

By Zenjiro KURAMOCHI

We shall extend some theorems of potential theory in space to abstract Riemann surfaces. In the present article we shall be concerned with Evans-Selberg's theorem on Riemann surfaces with null-boundary.

G. C. Evans and H. Selberg[2] proved the following theorem. *Given a closed set F of capacity zero in space, then there exists a positive mass distribution on F whose potential is positively infinite at every point of F.* We shall extend this theorem to abstract Riemann surfaces with null-boundary.

Let R^* be a Riemann surface with null-boundary and $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. Let $G_n(z, p)$ be the Green's function of R_n-R_0 with pole at p. Clearly, $G_n(z, p)\uparrow G(z, p)$ as $n\to\infty$. Since $\int_{\partial R_0}\frac{\partial G_n(z, p)}{\partial n}ds\leqq 2\pi$ for every n, $G(z, p)$ is not constant infinity and harmonic in R except at p where $G(z, p)$ has a logarithmic singularity.

Take M large so that the set $V_M(p)=E[z\in R: G(z, p)\geqq M]$ is compact in R. Let $\omega_n(z)$ be a harmonic function in $R_n-R_0-V_M(p)$ such that $\omega_n(z)=0$ on $\partial R_0+\partial V_M(p)$ and $\omega_n(z)=M$ on ∂R_n. Then since R^* is a Riemann surface with null-boundary, $\lim\limits_{n=\infty}\omega_n(z)=0$. Let $\bar{G}_n(z, p)$, $G_n'(z, p)$ and $\underline{G}_n(z, p)$ be harmonic functions in $R_n-R_0-V_M(p)$ such that $\bar{G}_n(z, p)=G_n'(z, p)=\underline{G}_n(z, p)=M$ on $\partial V_M(p)$, $\bar{G}_n(z, p)=G_n'(z, p)=\underline{G}_n(z, p)=0$ on ∂R_0 and $\bar{G}_n(z, p)=M$, $\frac{\partial G_n'(z, p)}{\partial n}=0$ and $\underline{G}_n(z, p)=0$ on ∂R_n respectively. Since $0<G_n'(z, p)<M$ on ∂R_n, we have by the maximum principle

$$\underline{G}_n(z, p)<G_n'(z, p)<\bar{G}_n(z, p),\quad \underline{G}_n(z, p)<G(z, p)<\bar{G}_n(z, p)$$

and

$$0<\bar{G}_n(z, p)-\underline{G}_n(z, p)=M\omega_n(z).$$

1) Resumé of this part is reported in Proc. Japan Acad. 32, 1956.

2) G. C. Evans: Potential and positively infinite singularities of harmonic functions. Monatsch. f. Math. u. Phys. 43, 1936, 419-424.

H. Selberg: Über die ebenen Punktmengen von der Kapazität Null. Avh. Norske Vid-akad, Oslo, 1, Nr. 10, 1937, 1-10.