Osaka Mathematical Journal
Vol. 8, No. 1, June, 1956.

One-to-one Continuous Mappings on Locally Compact Spaces

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It is a classical theorem of set-theoretical topology that a one-to-one continuous mapping φ of a bicompact Hausdorff space X onto a Hausdorff space Y is a homeomorphism[1]. But, in general, it cannot be said that such a mapping φ is homeomorphic if the spaces X and Y are both locally compact. In this paper we consider the problem for locally compact spaces with some special conditions.

Throughout this paper, we shall use the word "space" for "Hausdorff space". Let X and Y be two spaces, and let Y be a one-to-one continuous image of X under a mapping φ. If the inverse mapping φ^{-1} is not continuous, there exists a point x in X and a neighborhood U of x such that $\varphi(U)$ is not a neighborhood of $\varphi(x)$ in Y. Let D_φ denote the set of all such points x in X, i.e., D_φ is the set of all points x in X such that the inverse mapping $\varphi^{-1}(y)$ are not continuous at the points $\varphi(x)$. If D_φ is empty, the mapping φ is a homeomorphism from X to Y. When X is locally compact, a subset of X will be said to be bounded if the subset is contained in a bicompact subset of X. A° denotes always the interior of A for any subset A in X.

We shall prove the following theorem.

Theorem. *Let X be a locally compact Hausdorff space, and let X be represented as a union $\sum_{i=1}^{\infty} X_i$, where for each i X_i is bicompact, $X_i \subset X_{i+1}^\circ$ and $X-X_i$ is connected. Let Y be a locally compact but not bicompact and Hausdorff space, and let it be a one-to-one continuous image of X under φ. If the set D_φ is bounded in X, then the mapping φ is a homeomorphism.*

We shall prove first the following lemmas.

Lemma 1. *Let X be a locally compact space, and let φ be a one-to-one continuous mapping from X onto a space Y. Then the image $\varphi(D_\varphi)$ of D_φ under φ is closed in Y, and therefore D_φ is closed in X.*

Proof. Suppose that $\varphi(D_\varphi)$ is not closed in Y. Then there exists

1) See, for example, [1] p. 95, Satz III. Numbers in brackets refer to the references cited at the end of the paper.