Osaka Mathematical Journal
Vol. 8 No. 1, June, 1956.

On Quotient Rings

By Yuzo UTUMI

An extension ring S of a ring T is called a left quotient ring of T if for any two elements $x \neq 0$ and y of S there exists an element a of T such that $ax \neq 0$ and ay belongs to T. Let R be a ring without total right zero divisors. Then R has always a unique maximal left quotient ring, and moreover the maximal left quotient ring of a total matrix ring of finite degree over R is a total matrix ring of the same degree over the maximal left quotient ring of R.

A left ideal $\mathfrak{l}$ of R is called an M-ideal if it contains every element x for which there exists a left ideal $\mathfrak{m}$ of R satisfying the condition that (1) $\mathfrak{m}x \subseteq \mathfrak{l}$ and (2) R is a left quotient ring of $\mathfrak{m}$. When S is a left quotient ring of R, M-ideals of R and those of S correspond one-one in a definite way. A left ideal $\mathfrak{l}$ of R is said to be complemented if there exists a left ideal $\mathfrak{l}'$ such that $\mathfrak{l}$ is a maximal one among left ideals which have zero intersection with $\mathfrak{l}'$. Every complemented left ideal is an M-ideal, but the converse is not true in general. In a ring without total right zero divisors, every M-ideal is complemented if and only if the ring has the zero left singular ideal. Another example of M-ideals is the annihilator left ideals. A sufficient condition for that every M-ideal of a ring with zero left singular ideal is an annihilator left ideal, is that the maximal left quotient ring coincides with the maximal right quotient ring.

Every semisimple I-ring has zero singular ideals and hence it has the left and the right maximal quotient rings. We discuss especially two types of semisimple I-rings, i.e., primitive rings with nonzero socle, and semisimple weakly reducible rings. Let P be a primitive ring with nonzero socle. Then the maximal left quotient ring of P is right completely primitive. Thus, the left and the right maximal quotient rings of P coincide if and only if P satisfies the minimum condition. Let W be a semisimple weakly reducible ring. The left and the right maximal quotient rings of W always coincide and is also semisimple weakly reducible. In particular, if W is plain then its maximal quotient ring is strongly regular. This implies that the (nilpotency) index of a total matrix ring of degree m over a semisimple I-ring of index n is mn.