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On Principally Linear Elliptic Differential Equations of the Second Order.

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§ 0 Introduction

We use the notations $\underset{x_i}{\partial} u$ or $\underset{i}{\partial} u$ for $\dfrac{\partial u}{\partial x_i}$ and $\underset{x_i x_j}{\partial^2} u$ or $\underset{ij}{\partial^2} u$ for $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$. We write x for $x_1, \cdots, x_m$, $\underset{x}{\partial} u$ for $\underset{x_1}{\partial} u, \cdots \underset{x_m}{\partial} u$, and $\underset{x}{\partial^2} u$ for $\underset{ij}{\partial^2} u$ $(i, j = 1, \cdots, m)$.

In this note we shall consider principally linear partial differential equation[1] of elliptic type

$$\sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} u = f(x, u, \underset{x}{\partial} u). \tag{0}$$

We assume once for all that the quadratic form $\sum_{i,j=1}^{m} a_{ij}(x)\xi_i \xi_j$ *is positive definite.* We denote by $C[A]$ the set of all continuous functions on A, and by $C^p[A]$ the set of all functions whose partial derivatives up to the p-th order are all continuous on A. Under a solution of (0) in the domain D we understand a function of $C^2[D]$ which satisfies (0) for $x \in D$.[2] We say that a solution $u(x)$ of (0) in D takes the boundary value $\beta(x)$, when $u(x) \in C[\bar{D}]$ and $u(x) = \beta(x)$ for $x \in \dot{D}$.[3]

We say a function $\omega(x)$ is a *quasi-supersolution* (*-subsolution*) of (0) in a domain D, if for every point $x_0 \in D$, there exist a neighborhood U of x_0 and a finite number of functions $\omega_\nu(x) \in C^2[U]$ $(\nu = 1, \cdots, n)$ such that

$$\omega(x) = \operatorname*{Min}_{1 \leqq \nu \leqq n} \omega_\nu(x) \ \ (\operatorname*{Max}_{1 \leqq \nu \leqq n} \omega_\nu(x)) \qquad \text{for} \quad x \in U \tag{0.1}$$

and

$$\sum_{i,j=1}^{m} a_{ij}(x) \underset{ij}{\partial^2} \omega_\nu \leqq f(x, \omega_\nu, \underset{x}{\partial} \omega_\nu) \ (\geqq f(x, \omega_\nu, \underset{x}{\partial} \omega_\nu)). \tag{0.2}$$

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1) We say that a partial differential equation is principally linear, if it is linear in the terms of the highest derivatives with coeficients containing only independent variables.

2) D is a connected open set in the m-dimensional Euclidean space.

3) $\bar{D}$ means the closure of D, and $\dot{D}$ the boundary of D.