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Unbiasedness in the Test of Goodness of Fit

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1. Introduction. Let $X_1, \dots, X_N$ be a random sample from the population with the *d.f.* $F(x)$. We are asked to test the hypothesis H_0 that $F(x)$ is identical with a specified continuous *d.f.* $F_0(x)$ against all alternatives. For this purpose we shall use the multinomial distribution, dividing the real line into n intervals $(a_{i-1}, a_i]$, $i = 1, \dots, n$, where $a_0 = -\infty$ and $a_n = +\infty$, so that $F_0(a_i) - F_0(a_{i-1}) = 1/n$, $i = 1, \dots, n$. If a_i are not determined uniquely, we may take any values satisfying the conditions. Put $p_i = F(a_i) - F(a_{i-1})$ and denote by N_i the number of X's that fall into the interval $(a_{i-1}, a_i]$. Then, of course, $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n N_i = N$. Denote, further, by W the space consisting of n-dimensional lattice points $(k_1, \dots, k_n)$, where k_i is regarded as the observed value of the random variable N_i (therefore, $\sum_{i=1}^n k_i = N$).

The test is equivalent with determining the set (acceptance region) in the space W. The set S in W will be called symmetric provided that, if S contains the point $(k_1, \dots, k_n)$, then S contains also all its permutations $(k_1', \dots, k_n')$. We shall say, finally, that S satisfies condition O when, if S contains $(k_1, \dots, k_n)$ such as $k_j \geq k_i + 2$, then S contains also $(k_1, \dots, k_i + 1, \dots, k_j - 1, \dots, k_n)$. It is easily verified that if S is symmetric the convexity implies the condition O. The converse, however, is not necessarily true. For example, we shall consider, in the case $N = 12$, $n = 3$, the set S consisting of nine points shown in Fig. 1 and their permutations. S is symmetric and satisfies the condition O, but is not convex, since the middle point $(7, 4, 1)$ of the points $(8, 2, 2)$, $(6, 6, 0)$ does not belong to S.

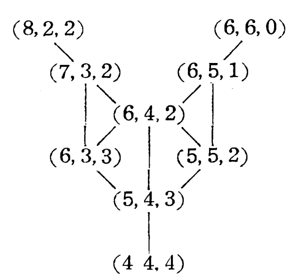

Fig. 1

2. Theorem of unbiasedness.

Theorem. *If the acceptance region R of the test is symmetric and satisfies the condition O, the test of H_0 is unbiased against any alternative.*

Proof. Putting