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A Characterization of Quasi-Frobenius Rings

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In this note we shall consider the problem: in what ring A can every homomorphism between two left ideals be extended to a homomorphism of A? ("Homomorphism" means "operator homomorphism"). We shall call this condition as *Shoda's condition.*[1] When A is a ring with a unit element, Shoda's condition is equivalent to the next one:

(α): *every homomorphism between two left ideals is given by the right multiplication of an element of A.*

The main purpose of this note is to show that if A is a ring with a unit element satisfying the minimum condition for left and right ideals, then A satisfies Shoda's condition if and only if A is a quasi-Frobenius ring.

T. Nakayama characterized quasi-Frobenius rings as the rings in which the duality relations $l(r(\mathfrak{l}))=\mathfrak{l}$ and $r(l(\mathfrak{r}))=\mathfrak{r}$ hold for every left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$.[2] Our result gives another characterization of quasi-Frobenius rings.

A denotes always a ring with the minimum condition for left and right ideals. Let N be the radical of A and $\bar{A}=A/N=\bar{A}_1+\cdots+\bar{A}_n$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals. Then, as is well known, we have two direct decompositions of A:

$$A=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}+l(E)=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}A+r(E) \tag{1}$$

where $E=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}$, $e_{\kappa,i}\,(\kappa=1,2,\cdots,n;\ i=1,2,\cdots f(\kappa))$ are mutually orthogonal primitive idempotents, $Ae_{\kappa,i}\cong Ae_{\kappa,1}=Ae_\kappa$ for $i=1,\cdots,f(\kappa)$, $Ae_{\kappa,i}\not\cong Ae_{\lambda,j}$ if $\kappa\neq\lambda$ and the same is true for $e_{\kappa,i}A$, and $l(*)$ $(r(*))$ is the left annihilator (right annihilator) of $*$. Moreover we use matric units $c_{\kappa,i,j}\,(\kappa=1,\cdots,n;\ i,j=1,\cdots,f(\kappa))$, $c_{\kappa,1,1}=e_{\kappa,1}=e_\kappa$, $c_{\kappa,i,i}=e_{\kappa,i}$ and $c_{\kappa,i,j}c_{\lambda,k,l}=\delta_{\kappa,\lambda}\delta_{j,k}c_{\kappa,i,l}$.

We start with the following preliminary lemmas.

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1) This problem was suggested by Prof. K. Shoda. Cf. K. Shoda [4].

2) See T. Nakayama [1], [2].