Osaka Mathematical Journal
Vol. 4, No. 2, December, 1952

On the Classification of Open Riemann Surfaces

By Yukinari Tôki

Introduction

We shall denote by O_{HB} and O_{HD} the classes of Riemann surfaces for which any single-valued harmonic functions that are respectively bounded or of finite Dirichlet integrals must be reduced to constants; furthermore we shall denote by O_G the class of Rieman surfaces without Green's functions. Then $O_G \leq O_{HB}$ was proved by P. J. Myrberg[1] and $O_{HB} \leq O_{HD}$ by Virtanen.[2] Recently Ahlfors gave an example[3] to prove that the first inclusion ($O_G < O_{HB}$) is strict. But unfortunately we can show his proof fails to prove this fact.

It was M. Inoue who pointed out for the first time that there is some vague point in Ahlfors' reasoning. He remarked: it is not always possible to conclude unconditionally that $\left(r\dfrac{\partial u}{\partial r}\right)^2$ is subharmonic at the end-points of the concentric circular slits.[4]

We shall show in § **1** that there exists a non-constant single-valued bounded harmonic function of Ahlfors' Riemann surface, in § 2 that Ahlfors' anticipation is right, by constructing a Riemann surface without non-constant single-valued bounded harmonic function but with the Green's function, and in § **3** that Virtanen's inclusion is indeed strict ($O_{HB} < O_{HD}$) by means of an example. I owe this investigation to Ahlfors' paper above mentioned.

For convenience we introduce some definitions. Let $\boldsymbol{D}$ be a domain in the z-plane and $\boldsymbol{F}$ a covering surface over the basic surface $\boldsymbol{D}$. Then, in determining the metric on $\boldsymbol{F}$ by that on $\boldsymbol{D}$, two cases can occur according as the sense of argument: for the mapping $\boldsymbol{F} \to \boldsymbol{D}$ the positive sense of argument on $\boldsymbol{F}$ is defined either

1) P. J. Myberg, Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche. Acta math. 61 (1933).

2) K. I. Virtanen, Ueber die Existenz von beschrankten harmonischen Funktionen auf offenen Riemannschen. Flächen. Ann. Acad. Scient. Fenn. A I 75 (1950).

3) L. v. Ahlfors, Remarks on the classification of open Riemann surfaces. Ann. Acad. Sic. Fenn. A. I. 87 (1951).

4) Ahlfors himself has recognized the defect of his proof. Cf. Math. Rev. vol. 13, No. 4, p. 338 (1952).