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Note on Brauer's Theorem of Simple Groups

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Using the theory of modular representations of groups, R. Brauer studied simple groups and obtained very interesting results[1] concerning a group which satisfies the following conditions:

($*$) *The group* $\mathfrak{G}$ *contains* P *of prime order* p *which commute only with their own powers* P^i.

($**$) *The commutator-subgroup* $\mathfrak{G}'$ *of* $\mathfrak{G}$ *is equal to* $\mathfrak{G}$.

By relaxing his conditions about the number of p-Sylow subgroups, we have the following theorem:

Theorem. *Let* $\mathfrak{G}$ *be a group of finite order which satisfies conditions* ($*$) *and* ($**$). *Then* $g = p(p-1)(1+np)/t$ *is the order of* $\mathfrak{G}$, *where* $1+np$ *is the number of conjugate subgroups of order* p *and* t *is the number of classes of conjugate elements of order* p *in* $\mathfrak{G}$. *If* $n < p+2$ *and* t *is odd, then* p *is of the form* $2^{\mu}-1$ *and* $\mathfrak{G} \cong LF(2,2^{\mu})$.

It seems probable that the case $\mathfrak{G} \cong LF(3,3)$ will occur, when t is even. But it is still an open problem.

Brauer mentioned in his earlier paper[2] that if $\mathfrak{G}$ is a simple group of order $g = qp(1+np)$ with $q|p-1$ in which the elements of order p commute only with their own powers and if $n < (2p+7)/3$, then either (1) $\mathfrak{G}$ is cyclic, or (2) $\mathfrak{G} \cong LF(2, p)$ or (3) p is a prime of the form $p = 2^{\mu} \pm 1$, and $\mathfrak{G} \cong LF(2,2^{\mu})$. (We can easily prove these facts by the slight modifications of his method).[3]

1. Preliminaries.

The former part of the theorem is obvious, so we shall prove only the latter half. In this paper we shall use the same notations as Brauer's and prove the theorem step by step with a little complicating numerical calculations.

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1) R. Brauer, On permutation groups of prime degree and related classes of groups, Ann. of Math. 44 (1943), I refer to this paper as [1].

2) R. Brauer, On the representation of groups of finite order, Proc. Nat. Akad. Sci. 25 (1939).

3) Cf. the proof of [1], Theorem 10. In the proof of Lemma 8 of this paper, we shall show the outline of them.