Osaka Mathematical Journal
Vol. 4, No. 1, May, 1952

Analytical Derivation of Sampling Distribution of Intraclass Correlation Coefficient

By Junjiro OGAWA

The exact sampling distribution of the intraclass correlation coefficient was obtained by R. A. Fisher in 1921.[1] However, his method of derivation is geometrical and too intuitive, so that it is not always easy for every one to comprehend the proof, at least so far as the writer knows. For this reason, it seems to me that, at least from the pedagogical point of view, the analytial derivation of sampling distribution of intraclass correlation coefficient is not at all valueless.

We shall first consider the case when the number of individuals belonging to the family is two. The probability element of the population distribution is given as follows:

$$df=\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\,e^{-\frac{1}{2\sigma^2(1-\rho^2)}\left\{(x-m)^2-2\rho(x-m)(x'-m)+(x'-m)^2\right\}}\,dx\,dx'\,.$$

where m, σ, and ρ are the mean, standard deviation and intraclass correlation coefficient of the population respectively. The necessary statistics will be defined as follows:

$$nm_{20}=\sum_{i=1}^{n}(x_i-\bar{x})^2\,,\ nm_{02}=\sum_{i=1}^{n}(x_i'-\bar{x}')^2\,,\ nm_{11}=\sum_{i=1}^{n}(x_i-\bar{x})(x_i'-\bar{x}')\,,$$

$$n\bar{x}=\sum_{i=1}^{n}x_i\,,\ n\bar{x}'=\sum_{i=1}^{n}x_i'\,.$$

$$2n\bar{\bar{x}}=\sum_{i=1}^{n}(x_i+x_i')\,.$$

$$2n\mu^2=\sum_{i=1}^{n}\left\{(x_i-\bar{\bar{x}})^2+(x_i'-\bar{\bar{x}})^2\right\}\,,$$

and

$$n\mu^2 r=\sum_{i=1}^{n}(x_i-\bar{\bar{x}})(x'-\bar{\bar{x}})\,,$$

where r is the intraclass correlation coefficient of the sample and our purpose is to obtain the exact sampling distribution of r.

It is readily seen that