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Linear-Order on a Group

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Let us discuss here under what condition a group admits a linear-order.[1] Related ideas to my previous paper[2] will be adopted.

Preliminaries. A partial-order on a group G is wholly determined by such a subset $\mathfrak{g}$ of G — we shall call it a (partial-) *ordering set* briefly — that satisfies the following two conditions:

1) $\mathfrak{g}$ is an invariant sub-semigroup with 1,

2) $\mathfrak{g}$ cannot contain an element ($\neq 1$) together with its inverse.

A linear-ordering set is therefore characterized by one more additional condition:

3) It contains either x or x^{-1} for any x of G.

For a finite subset $\{x_1, \dots, x_n\}$ of G and an invariant sub-semigroup $\mathfrak{g}$ of G the invariant sub-semigroup generated by $x_1, \dots, x_n$ and $\mathfrak{g}$ shall be denoted by $\mathfrak{g}(x_1, \dots, x_n)$. Especially the invariant sub-semigroup generated by $\{x_1, \dots, x_n\}$ alone is $(x_1, \dots, x_n)$.

Theorem. *The following three conditions are mutually equivalent:*

(I) *G admits a linear-order.*

(II) *For any finite subset $\{x_1, \dots, x_n\}$ of G the intersection of all possible 2^n $\mathfrak{g}(x_1^{\varepsilon_1}, \dots x_n^{\varepsilon_n})$, where $\varepsilon_i = \pm 1$, is equal to 1.*

(III) *For any element a of G there exists an ordering set $\mathfrak{g}_a$ containing a and having the property:*

(*) *If xy ($\neq 1$) belongs to $\mathfrak{g}_a$, then either x or y belongs to $\mathfrak{g}_a$.*

Such an ordering set in (III) will be called (*)-ordering set.

Proof. We shall divide this into three parts:

(I)→(II). By a linear-order on G every element x of G attains a sign $\varepsilon^0 = \pm 1$ in such a way that x^{ε_0} is $\geqq 1$ with respect to this order. Then obviously all elements of $(x_1^{\varepsilon_1^0}, \dots, x_n^{\varepsilon_n^0})$ are $\geqq 1$, and all elements of $(x_1^{-\varepsilon_1^0}, \dots, x_n^{-\varepsilon_n^0})$ are $\leqq 1$. Therefore the intersection of these two sets is already equal to 1.

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1) Cf. K. Iwasawa, On linearly ordered groups. Journ. of Math. Soc. of Japan, 1 (1948).

Also, P. Lorenzen, Ueber halbgeordnete Gruppen. Math. Zeits. 52 (1949).

2) M. Ohnishi, On linearization of ordered groups. Osaka Math. Journ. 2 (1950).