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On Cartesian Product of Compact Spaces

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While the Cartesian product of any number of compact (= bicompact) spaces is again compact by Tychonoff's theorem [1], there is an $\aleph_0$-compact (= compact in the sense of Fréchet) space R whose product $R\times R$ is not $\aleph_0$-compact,[1] as will be shown in the present note. These circumstances will be somewhat clarified by the introduction of a concept of $\aleph_\alpha$-ultracompactness.

1. Let M be a given set of points and let $\boldsymbol{M}=\{M_\lambda\}$ be an ultrafilter [2], i.e., a collection of subsets M_λ of M such that

(i) $\boldsymbol{M}$ has the finite intersection property, i.e., any finite number of M_λ's have a non-void intersection,

(ii) $\boldsymbol{M}$ is maximal with respect to the property (i), i.e., should any subset M' of M distinct from any one of M_λ be added to $\boldsymbol{M}$, then the resulting collection $\boldsymbol{M}+M'$ fails to satisfy the condition (i).

If $\aleph_\alpha$ denotes the lowest of the potencies of M_λ, we say that $\boldsymbol{M}$ is of *potency* $\aleph_\alpha$. A T_1-space will be called *$\aleph_\alpha$-ultracompact*, if every ultrafilter of potency $\aleph_\alpha$ has a cluster point. Then the proof of C. Chevalley and O. Frink [3] for Tychonoff's theorem yields at once the following

Theorem. *The Cartesian product of any nnmber of $\aleph_\alpha$-ultracompact spaces is itself $\aleph_\alpha$-ultracompact.*

Here arises the question, *whether or not, if R is $\aleph_\alpha$-compact, i.e., if every subset $M\subset R$ of potency $\aleph_\alpha$ has a cluster point, but if R is not $\aleph_\alpha$-ultracompact, then the product* ΠR *is not necessarily $\aleph_\alpha$-compact.* As a partly solution of this question we construct in the following an example of an $\aleph_0$-compact but not $\aleph_0$-ultracompact space R, whose product $R\times R$ is not $\aleph_0$-compact.

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1) The question whether or not such an $\aleph_0$-compact space exists was raised by M. Ohnishi of Osaka University and answered by me in Sizyo Sugaku Danwakai (June 10, 1947): An example of an $\aleph_0$-compact space R whose product $R\times R$ is not $\aleph_0$-compact (In Japanese). After I had written the present note I have been informed by Ohnishi that the question is originally that of Čech, for which an answer is announced to have been given by Novák in Časopis propěst. mat. a fys. 74 (1950).