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Some Remarks on Unitary Representations of the Free Group

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§ 1. INTRODUCTION. This paper contains three (at least superficially) independent remarks concerning "pathological" phenomena appearing in the theory of unitary representations.[1] In order to show the phenomena in their extreme forms and to simplify arguments, we make the examples of the free group of two generators, though some of these phenomena are not peculiar to the free group and may be more significant and interesting in cases of more "regular" groups.

In § 2 we shall consider the following property:

(K) The constant 1 is uniformly approximated on compact by the convolution of function belonging to $L^2(G)$ with its adjoint.[2]

As shown by Godement,[2] for a unimodular (= with a two-sided invariant measure) group, (K) is equivalent with the property that every positive definite function is similarly approximated or that the integral of an absolutely integrable positive definite function is non-negative. It is of important meaning in the theory of unitary representations,[3] but recently it is known to be valid only in a rather special class of locally compact groups among general ones. In particular, the property (K) is very simply disproved for the free group.[4]

In § 3 it will be shown for the free group that, in the set $\boldsymbol{P}_1$ of

1) For the matters connected with those of §§ 2 and 3 of this paper, which were studied during 1947-8, the athour is much indebted to Professor Kakutani, who, however, is in no way related to the present paper itself which is written in 1950.

2) See A. Weil's book "L'intégration dans les groupes topologiques et ses applications" and R. Godement: *Les fonctions de type positif et la théorie des groupes*, Trans. Amer. Math. Soc., **63**. A. Khintchine proved it for the case of the real number group.

3) Moreover from it follow some interesting properties of the group, e.g., the symmetricity of the L^1-algebra and the existence of an invariant mean for bounded continuous functions.

4) The author obtained these results in the winter of 1947-8, but afterwards he was informed that essentially the same result as in § 2 had been published (by a lecture) already in 1944 by Professor Kakutani.