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On Sufficient Conditions for a Function to be Holomorphic in a Domain

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§ 1

1. The problem under what condition it is sufficient for the continuous function $f(z)=U(z)+iV(z)$ of a complex variable $z=x+iy$ defined in a domain D of the z-plane to be holomorphic, has been studied from many points of view. In particular one is from the theory of a real function or the integral, and the other is from the properties of an analytic function in the neighbourhood of the regular point, for instance, the invariance of segment's ratio, of angles, etc. The latter is the starting point of Menchoff's study continued from 1923 to 1938.

In regarding this there may be enumerable algebraic singular points (i. e. branch point) at which the local properties in the neighbourhood will be lost to some extent, his allowance that there might be enumerable points at which the properties supposed as the conditions of his theorems, were not satisfied, renders to be more interesting in the case when $f(z)$ is not univalent, because univalent and holomorphic function cannot have any branch points in its domain. The object of our study is to extend his theorems so as they may remain valid even when $f(z)$ is not necessarily univalent, to shorten his proofs and generalize in some ways.

When $\lim\frac{f(z_0+h)-f(z_0)}{h}$ exists, we call $f(z)$ is monogene at $z=z_0$. The necessary and sufficient conditions for $f(z)$ to be monogene, is that $f(z)$ is totally derivable [1] and simultaneously satisfies the Cauchy-Riemann differential equations $\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}$, $\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}$ and the necessary and sufficient conditions for $f(z)$ to be holomorphic in D is that $f(z)$ is monogene at every point in D. We see directly that the set in which $f(z)$ is not regular forms a perfect set.

2. We denote the half lines issuing from z by $\tau_i(z)$; $i=1.2.3...,$

1) Pompeiu : Sur la continuité des fonctions de varibles complexes, Ann. Fac. Soc. Université Toulouse (2), pp. 262-315 (1905).