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On Linearization of Ordered Groups

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In his recent paper, "Note on a result of L. Fuchs on ordered groups",[1] C. J. Everett has shown: any partial order on an *abelian* group G can be extended into a linear one, if (and trivially only if) every element of G except the unit is of infinite order.

Let us discuss here the non-abelian case in a similar way as Everett. By a *partial order on a group* G, we require that *this order should be preserved under the group operation*, i. e.

$$a > b \text{ implies } ax > bx \text{ and } xa > xb \text{ for all } x \text{ in } G.$$

Such a partial order on G is completely determined by the set $\mathfrak{P}$ of all elements p 1 (the unit) of G. $\mathfrak{P}$ has namely the following characterizing properties:

1) $\mathfrak{P}$ is a self-conjugate semi-group with 1,

2) $\mathfrak{P}$ contains no element along with its inverse except 1.

Now we are to enlarge this set $\mathfrak{P}$ until for every $x(\neq 1)$ in G either x or x^{-1} belongs to $\mathfrak{P}$, that is to extend the given partial order into a linear one.

First we need some preliminary notations and remarks. Let

$$C_a = \{xax^{-1};\ x \in G\},$$

and further

$$\mathfrak{C}_a = \textstyle\sum_{n=1}^{\infty} (C_a)^n,$$

where $(C_a)^n$ denotes the set of all elements of the form $a_1 a_2 \dots a_n$, $a_i \in C_a$, and $\sum$ means the set-union.

If G admits a linear order at all, then

(I) *$\mathfrak{C}_a$ and $\mathfrak{C}_a^{-1}$ are disjoint for every a.*

Moreover, calling two elements a and b *equivalent*, if there exists a finite chain $\mathfrak{C}_{a_1}, \mathfrak{C}_{a_2}, \dots, \mathfrak{C}_{a_k}$, where $a \in \mathfrak{C}_{a_1}$, $\mathfrak{C}_{a_1} \wedge \mathfrak{C}_{a_2} \neq 0$, $\mathfrak{C}_{a_2} \wedge \mathfrak{C}_{a_3} \neq 0$, $\dots$, $\mathfrak{C}_{a_{k-1}} \wedge \mathfrak{C}_{a_k} \neq 0$, $\mathfrak{C}_{a_k} \ni b$, we easily see that this equivalence satisfies the usual relations of equivalence, and we get the following necessary

1): Amer. J. Math. 72, p. 216 (1950).