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Mixing Up Property of Brownian Motion.

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1. Let X be the space of Brownian motions. Any element x of X is to be considered here as a set function defined for intervals on the infinite line. We denote the value of x for the interval (a, b) by $x(b)-x(a)$, but it is to be noticed that $x(a)$ or $x(b)$ alone has no meaning in our case. Let $\varphi_t\,(-\infty< t<\infty)$ be the flow of translations on X defined by $\varphi_t x(a, b)=x(a+t, b+t)$, that is, $x'=\varphi_t x$ means that $x'(b)-x'(a)=x(b+t)-x(a+t)$. It is well known that φ_t is strongly mixing.[1]

THEOREM 1. *Let X be the space of Brownian motions as set functions and let φ_t be the flow of translations on X above defined. Then for any measurable ergodic flow ψ_t on a measure space Y, the skew product flow T_t, which is defined in the following way on the direct product measure space Ω of X and Y, is strongly mixing.*

$$T_t(x, y)=(\varphi_t x, \psi_{x(t)-x(0)}\, y),\ \textit{where}\ (x, y)\in\Omega=X\times Y.$$

At first, it is easily verified that for any fixed t, T_t is a measure preserving transformation on Ω, from the fact that for any null set N in the usual Lebesgue measure space of the infinite line, the set $\{x\,|\,x(t)-x(0)\in N\}$ is a null set in X.[2] And further we may show in the same way that T_t is a measurable flow on Ω from the fact that the (t, x)-set $\{(t, x)\,|\,x(t)-x(0)\in N\}$ is a set of (t, x)-measure zero.[2] In order to prove that T_t is strongly mixing, it is sufficient to show that

$$\text{(1)}\qquad \lim_{t\to\infty}\iint f(\varphi_t x)\, f_1(\psi_{x(t)-x(0)}y)\, g(x)\, g_1(y)\, dxdy$$
$$=\iint f(x)\, f_1(y)\, dxdy \iint g(x)\, g_1(y)\, dxdy$$

holds for any $f(x)$, $g(x)\in L^2(X)$ and $f_1(y)$, $g_1(y)\in L^2(Y)$.

Let E_λ be the spectral resolution of the one-parameter group of unitary transformations of $L^2(Y)$, which corresponds to the flow ψ_t on Y, and put $v(\lambda)=(E_\lambda f_1, g_1)$. Then the left hand side of (1) is equal to

1) E. HOPF, Ergodentheorie, Berlin, 1937, p. 59 §16. Masstheorie im Raum der additiven Mengenfunktionen. Das Spektrum der Translationen.

2) See (7) of **2** of this note.